Numerical Analysis of PDEs **F10ND2/F11ND2**

Exam 2013 - Solutions

Question 1. [50 Marks total]

(a) By definition,

$$B_x u(x) = u(x) - u(x - \Delta x), \qquad \delta_x^2 u(x) = u(x + \Delta x) - 2u(x) + u(x - \Delta x);$$

Expanding $u(x \pm \Delta x)$ in a Taylor series, we have

$$u(x \pm \Delta x) = u \pm \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} \pm \frac{1}{3!}\Delta x^3 u_{xxx} + \frac{1}{4!}\Delta x^2 u_{xxxx} + \dots, \Big|_x$$

so

$$\begin{aligned} \frac{B_x}{\Delta x} u &= \frac{1}{\Delta x}\left(u - \left(u - \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} - \frac{1}{3!}\Delta x^3 u_{xxx} + \dots\right)\right) \\ &= u_x - \frac{\Delta x}{2} u_{xx} + O(\Delta x^2) \end{aligned}$$

and

$$\begin{aligned} \frac{\delta_x^2}{\Delta x^2} u = \frac{1}{\Delta x^2}\Bigg[&\left(u + \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} + \frac{1}{3!}\Delta x^3 u_{xxx} + \frac{1}{4!}\Delta x^4 u_{xxxx} + \dots\right) - 2u + \\ &+ \left(u - \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} - \frac{1}{3!}\Delta x^3 u_{xxx} + \frac{1}{4!}\Delta x^4 u_{xxxx} + \dots\right)\Bigg], \\ = u_{xx} + \frac{\Delta x^2}{12} u_{xxxx} &+ O(\Delta x^4), \end{aligned}$$

as required.

(b) For the BTCS scheme

$$L_\Delta = \frac{B_t}{\Delta t} - \frac{\delta_x^2}{\Delta x^2}.$$

so the LTE is given by

$$\text{LTE} = L_\Delta u(x,t) = u_t - \frac{\Delta t}{2} u_{tt} + O(\Delta t^2) - \left(u_{xx} + \frac{\Delta x^2}{12} u_{xxxx} + O(\Delta x^4)\right).$$

Since $u_t = u_{xx}$ from the PDE, and by repeated differentiation $u_{tt} = u_{xxxx}$ we have after some simplification

$$\text{LTE} = -\left(\frac{1}{2}\Delta t + \frac{1}{12}\Delta x^2\right) u_{xxxx} + O(\Delta t^2) + O(\Delta x^4),$$

and hence $a = -1/2$ and $b = -1/12$.

We have

$$
\begin{aligned}
& L_\Delta w_j^{n+1} = 0 \\
\Rightarrow \quad & \frac{B_t}{\Delta t} w_j^{n+1} = \frac{\delta_x^2}{\Delta x^2} w_j^{n+1}, \\
\Rightarrow \quad & \frac{1}{\Delta t}(w_j^{n+1} - w_j^n) = \frac{1}{\Delta x^2}(w_{j-1}^{n+1} - 2w_j^{n+1} - w_{j+1}^{n+1}), \\
\Rightarrow \quad & -r(w_{j-1}^{n+1} + w_{j+1}^{n+1}) + (1+2r)w_j^{n+1} = w_j^n,
\end{aligned}
$$

for $j = 1, \ldots, J-1$ and $n \geq 0$ where $J = 1/\Delta x$ and $r = \Delta t/\Delta x^2$.

Testing for stability, put $w_j^n = \xi^n e^{i\omega j}$ into the scheme to get after cancellation of $\xi^n e^{i\omega j}$

$$\xi(1 + 2r - re^{i\omega} - re^{-i\omega}) = 1,$$

so

$$\xi = 1/(1 + 2r - 2r\cos(\omega)) = 1/(1 + 4r\sin^2(\omega/2)) \leq 1,$$

since both r and $\sin^2$ are positive. Clearly $\xi > 0$ for the same reason. So $|\xi| \leq 1$ for all r and the scheme is unconditionally stable.

(c) If $J = 4$ and $r = 0.5$ then $\Delta x = 1/4$ and $\Delta t = 1/32$. We have

$$\mathbf{w}^0 = [0.5\sin(\pi j \Delta x), j = 0, \ldots, 4] = [0, \frac{0.5}{\sqrt{2}}, 0.5, \frac{0.5}{\sqrt{2}}, 0] = [0, 0.353552, 0.5, 0.353552, 0].$$

Applying the scheme above for $j = 1, \ldots, 3$ gives

$$
\begin{aligned}
2w_1^1 - 0.5w_2^1 \qquad &= 0.5/\sqrt{2} = 0.353552, \\
-0.5w_1^1 + 2w_2^1 - 0.5w_3^1 &= 0.5, \\
-0.5w_2^1 + 2w_3^1 &= 0.5/\sqrt{2} = 0.353552.
\end{aligned}
$$

Question 2. [50 Marks total]

(a) The FTCS scheme is

$$\begin{aligned} \frac{F_t}{\Delta t} w_j^n &= \frac{\delta_x^2}{\Delta x^2} w_j^n - w_j^n \\ \Rightarrow \left(\frac{w_j^{n+1} - w_j^n}{\Delta t} \right) &= \left(\frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{\Delta x^2} \right) - w_j^n, \end{aligned}$$

and the BTCS scheme is

$$\begin{aligned} \frac{B_t}{\Delta t} w_j^{n+1} &= \frac{\delta_x^2}{\Delta x^2} w_j^{n+1} - w_j^{n+1} \\ \Rightarrow \left(\frac{w_j^{n+1} - w_j^n}{\Delta t} \right) &= \left(\frac{w_{j-1}^{n+1} - 2w_j^{n+1} + w_{j+1}^{n+1}}{\Delta x^2} \right) - w_j^{n+1}. \end{aligned}$$

Taking the average of these two gives

$$\begin{aligned} \left(\frac{w_j^{n+1} - w_j^n}{\Delta t} \right) &= \frac{1}{2} \left(\frac{\delta_x^2}{\Delta x^2} w_j^{n+1} + \frac{\delta_x^2}{\Delta x^2} w_j^n - (w_j^{n+1} + w_j^n) \right) \\ \Rightarrow \left(\frac{w_j^{n+1} - w_j^n}{\Delta t} \right) &= \left(\frac{w_{j-1}^{n+1} - 2w_j^{n+1} + w_{j+1}^{n+1}}{2\Delta x^2} \right) + \left(\frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{2\Delta x^2} \right) - \left(\frac{w_j^{n+1} + w_j^n}{2} \right). \end{aligned}$$

multiplying through by Δt, defining $r = \Delta t / \Delta x^2$, and collecting t_{n+1} terms on the left we have

$$-\frac{r}{2} w_{j-1}^{n+1} + \left(1 + r + \frac{\Delta t}{2} \right) w_j^{n+1} - \frac{r}{2} w_{j+1}^{n+1} = \frac{r}{2} w_{j-1}^n + \left(1 - r - \frac{\Delta t}{2} \right) w_j^n + \frac{r}{2} w_{j+1}^n, \quad j = 1, \ldots, J-1.$$

(b) when $J = 2$, $r = 1$, then $\Delta x = 0.5$ $\Delta t = 0.25$ and the above equations become

$$-0.5 w_{j-1}^{n+1} + 2.125 w_j^{n+1} - 0.5 w_{j+1}^{n+1} = 0.5 w_{j-1}^n - 0.125 w_j^n + 0.5 w_{j+1}^n, \quad j = 1.$$

We have $\mathbf{w}^0 = [1, 2, 1]$.

The equation for the w_1^1 are, after taking into account that $w_0^0 = w_2^0 = w_0^1 = w_2^1 = 1$,

$$2.125 w_1^1 = 2 - 0.125 w_1^0,$$

i.e.,

$$w_2^1 = 0.823529.$$

(c) Substitute $w_m^n = \xi^n e^{im\omega}$ and simplify in the usual way

$$\begin{aligned} &-\frac{1}{2} r e^{i(m-1)\omega} \xi^{n+1} + (1 + r + 0.5\Delta t) e^{im\omega} \xi^{n+1} - \frac{1}{2} r e^{i(m+1)\omega} \xi^{n+1} = \\ &\qquad \frac{1}{2} r e^{i(m-1)\omega} \xi^n + (1 - r - 0.5\Delta t) \xi^n + \frac{1}{2} r e^{i(m+1)\omega} \xi^n \\ &\Rightarrow -\frac{1}{2} r e^{-i\omega} \xi + (1 + r + 0.5\Delta t) \xi - \frac{1}{2} r e^{i\omega} \xi = \frac{1}{2} r e^{-i\omega} + (1 - r - 0.5\Delta t) + \frac{1}{2} r e^{i\omega} \\ &\qquad \text{(taking out factors } e^{im\omega} \text{ and } \xi^n) \\ &\Rightarrow (1 + 0.5\Delta t) \xi - \xi \frac{1}{2} r (e^{i\omega} - 2 + e^{-i\omega}) = 1 - 0.5\Delta t + \frac{1}{2} r (e^{i\omega} - 2 + e^{-i\omega}) \end{aligned}$$

Now

$$e^{i\omega} - 2 + e^{-i\omega} = -2(1 - \cos(\omega)) = -4\sin^2(\omega/2)$$

so the above becomes

$$(1 + 0.5\Delta t)\xi + 2\xi \sin^2(\omega/2) r = 1 - 0.5\Delta t - 2r\sin^2(\omega/2)$$
$$\Rightarrow \xi = \frac{1 - 0.5\Delta t - 2r\sin^2(\omega/2)}{1 + 0.5\Delta t + 2r\sin^2(\omega/2)}$$

We need $|\xi| \leq 1$ for stability for all $\omega \in [-\pi, \pi]$. Since ξ is clearly real in this case this means we require $-1 \leq \xi \leq 1$. Now

$$\xi = \frac{1 + 0.5\Delta t + 2r\sin^2(\omega/2) - \Delta t - 4r\sin^2(\omega/2)}{1 + 0.5\Delta t + 2r\sin^2(\omega/2)} = 1 - \frac{\Delta t + 4r\sin^2(\omega/2)}{1 + 0.5\Delta t + 2r\sin^2(\omega/2)}$$

so ξ is clearly always less than $+1$. Now consider the inequality $\xi \geq -1$. This is (on multiplying through by the denominator)

$$-1 - 0.5\Delta t - 2r\sin^2(\omega/2) \leq 1 - 0.5\Delta t - 2r\sin^2(\omega/2)$$
$$\Rightarrow -1 \leq 1$$

This last clearly holds for *all* r. Hence the inequality $|\xi| \leq 1$ is always true and the method is unconditionally stable.

Question 3. [50 Marks total]

(a) The upwind method is

$$\frac{w_j^{n+1}-w_j^n}{\Delta t}+a\frac{w_j^n-w_{j-1}^n}{\Delta x}=0.$$

The LTE of the scheme is

$$\begin{aligned}\text{LTE} &= \frac{u(x_j,t_{n+1})-u(x_j,t_n)}{\Delta t}+a\frac{u(x_j,t_n)-u(x_{j-1},t_n)}{\Delta x}\\ &= \left(u_t+\frac{1}{2}\Delta t u_{tt}+O(\Delta t^2)+a\left[u_x-\frac{1}{2}\Delta x u_{xx}+O(\Delta x^2)\right]\right)\Bigg|_{(x_j,t_n)}\\ &= \underbrace{(u_t+au_x)}_{=0\text{ by PDE}}+\underbrace{\frac{1}{2}(\Delta t\, u_{tt}-a\Delta x\, u_{xx})}_{\neq 0\text{ in general}}+O(\Delta t^2,\Delta x^2),\end{aligned}$$

hence, the leading term is $O(\Delta t,\Delta x)$ and the method is of first order.

(b) Inserting $w_j^n=\xi^n e^{i\omega j}$ into the L-W scheme and simplifying, we get

$$\begin{aligned}\xi &= (1-p^2)-\frac{1}{2}p(1-p)e^{i\omega}+\frac{1}{2}p(1+p)e^{-i\omega}\\ &= 1+p^2(\cos\omega-1)-ip\sin\omega\\ &= 1-2p^2\sin^2(\omega/2)-ip\sin\omega\\ &= 1-2p^2\sin^2(\omega/2)-2ip\sin(\omega/2)\cos(\omega/2)\end{aligned}$$

So

$$\begin{aligned}|\xi|^2 &= \left[1-2p^2s^2\right]^2+4p^2s^2c^2,\ \text{ where } s=\sin(\omega/2), c=\cos(\omega/2)\\ &= 1+4p^2s^2(c^2-1)+4p^4s^4\\ &= 1-4p^2(1-p^2)s^4\end{aligned}$$

Clearly this is ≤ 1 for all $|p|\leq 1$ and >1 for all $|p|>1$, so the scheme is stable if and only if $|p|\leq 1$.

(c) The upwind scheme is only first order in space and time and suffers from numerical diffusion (artificial viscosity). The Lax-Wendroff scheme is second order accurate which is better than the upwind scheme and produces less numerical diffusion, but the numerical solution can contain some oscillations. Both methods are stable for $|p|\leq 1$.

Question 4. Inserting the approximation

$$u(x) \approx w(x) \sum_{k=1}^{N} c_k \phi_k(x),$$

into the functional $J[u]$ we get

$$J[\mathbf{c}] = \frac{1}{2} \int_0^1 \left[\left(\sum_{k=1}^{N} c_k \frac{d\phi_k(x)}{dx} \right)^2 + \left(\sum_{k=1}^{N} c_k \phi_k(x) \right)^2 + 2 \sum_{k=1}^{N} c_k f(x) \phi_k(x) \right] dx$$

Now minimise over the c_k, for this we require that $\partial J / \partial c_j = 0,\ j = 1, \ldots, N$, so

$$\int_0^1 \left[\frac{d\phi_j}{dx} \sum_{k=1}^{N} c_k \frac{d\phi_k}{dx} + \phi_j \sum_{k=1}^{N} c_k \phi_k + f(x)\phi_j(x) \right] dx = 0$$

$$\Rightarrow \sum_{k=1}^{N} c_k \int_0^1 \left[\frac{d\phi_j}{dx} \frac{d\phi_k}{dx} + \phi_j \phi_k \right] dx + \int_0^1 f(x)\phi_j(x) dx = 0$$

$$\Rightarrow \sum_{k=1}^{N} a_{j,k} c_k + b_j = 0, \qquad \text{or} \quad A\mathbf{c} = -\mathbf{b}$$

where

$$a_{j,k} = \int_0^1 \left[\frac{d\phi_j}{dx} \frac{d\phi_k}{dx} + \phi_j \phi_k \right] dx, \quad b_j = \int_0^1 f(x)\phi_j(x).$$

For the basis functions $\phi_n(x)$ shown in the question, these are piecewise linear functions given by

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/(x_k - x_{k-1}), & x_{k-1} \le x \le x_k \\ (x_{k+1} - x)/(x_{k+1} - x_k), & x_k \le x \le x_{k+1} \\ 0, & \text{otherwise.} \end{cases}$$

Each $\phi_j(x)$ is nonzero over only two elements, $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, and takes the value 1 at $x = x_k$.

We can now calculate b_j easily ($f \equiv 1$)

$$\begin{aligned} b_j &= \int_0^1 f\phi_j \, dx = \int_{x_j - 1}^{x_j} (x - x_{j-1})/(x_j - x_{j-1}) \, dx + \int_{x_j}^{x_{j+1}} (x - x_{j+1})/(x_j - x_{j+1}) \, dx, \\ &= \frac{1}{2} \frac{(x - x_{j-1})^2}{(x_j - x_{j-1})} \Big|_{x=x_{j-1}}^{x=x_j} + \frac{1}{2} \frac{(x - x_{j+1})^2}{(x_j - x_{j+1})} \Big|_{x=x_j}^{x=x_{j+1}} = \frac{1}{2}(x_j - x_{j-1}) + \frac{1}{2}(x_{j+1} - x_j), \\ &= \Delta x, \end{aligned}$$

since the nodes are equally spaced, $x_j - x_{j-1} = x_{j+1} - x_j = \Delta x$. We could have by-passed the integration process by noting that the integral is just $f \times$ the area of a triangle with height 1 and base $2\Delta x$.

We note that ϕ_k' is a piecewise constant function, in the equally spaced case

$$\phi_k'(x) = \begin{cases} 1/\Delta x, & x_{k-1} \le x \le x_k \\ -1/\Delta x, & x_k \le x \le x_{k+1} \\ 0, & \text{otherwise.} \end{cases}$$

Now consider the matrix element $a_{k,k} = \int [\phi_k'(x)^2 + \phi_k(x)^2]\, dx$. The integrand is nonzero over both $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$. Now

$$\int \phi_k'(x)^2\, dx = \int_{x_{k-1}}^{x_k} \frac{1}{\Delta x^2}\, dx + \int_{x_k}^{x_{k+1}} \frac{1}{\Delta x^2}\, dx = \frac{2}{\Delta x},$$

and

$$\begin{aligned}\int \phi_k(x)^2\, dx &= \int_{x_{k-1}}^{x_k} \frac{(x - x_{k-1})^2}{\Delta x^2}\, dx + \int_{x_k}^{x_{k+1}} \frac{(x - x_{k+1})^2}{\Delta x^2}\, dx \\ &= \left[\frac{(x - x_{k-1})^3}{3\Delta x^2}\right]_{x_{k-1}}^{x_k} + \left[\frac{(x - x_{k+1})^3}{3\Delta x^2}\right]_{x_k}^{x_{k+1}} = \frac{2\Delta x}{3},\end{aligned}$$

Now consider $a_{k-1,k} = \int \phi_{k-1}'(x)\phi_k'(x) + \phi_{k-1}(x)\phi_k(x)\, dx$. The integrand is nonzero only over $[x_{k-1}, x_k]$. Now

$$\int \phi_{k-1}'(x)\phi_k'(x)\, dx = \int_{x_{k-1}}^{x_k} \frac{-1}{\Delta x} \cdot \frac{1}{\Delta x}\, dx = -\frac{1}{\Delta x}.$$

and

$$\begin{aligned}\int \phi_{k-1}(x)\phi_k(x)\, dx &= \int_{x_{k-1}}^{x_k} \frac{(x - x_{k-1})(x_k - x)}{\Delta x^2}\, dx = \int_{x_{k-1}}^{x_k} \frac{(x - x_{k-1})(x_{k-1} + \Delta x - x)}{\Delta x^2}\, dx \\ &= \int_{x_{k-1}}^{x_k} \frac{-(x - x_{k-1})^2 + \Delta x(x - x_{k-1})}{\Delta x^2}\, dx \\ &= \left.\frac{-(x - x_{k-1})^3/3 + \Delta x(x - x_{k-1})^2/2}{\Delta x^2}\right|_{x_{k-1}}^{x_k} = \frac{\Delta x}{6}.\end{aligned}$$

So finally we have

$$a_{k,k-1} = \frac{\Delta x}{6} - \frac{1}{\Delta x}, \quad a_{k,k} = \frac{2}{\Delta x} + \frac{2\Delta x}{3}, \text{ and } b_k = h.$$