Question 1.

The heat equation $u_t = u_{xx}$ for $u(x,t)$ with $x \in [0,1]$ is to be approximated on a uniform rectangular grid with mesh spacing $\Delta x = 1/J$. The boundary conditions are $u(0,t) = 0, u(1,t) = 0$ and the initial condition is $u(x,0) = \frac{1}{2}\sin(\pi x)$. The time step is Δt and the mesh ratio is $r = \Delta t/\Delta x^2$. The approximate solution is $w_j^n \approx u(j\Delta x, n\Delta t) = u(x_j, t_n)$.

(a) Define the backward (B_x) and second central (δ_x^2) difference operators acting on the smooth function $u(x,t)$ in terms of the mesh spacing Δx. Show that

$$\frac{B_x}{\Delta x}u = u_x - \frac{\Delta x}{2}u_{xx} + O(\Delta x^2) \qquad \text{and} \qquad \frac{\delta_x^2}{\Delta x^2}u = u_{xx} + \frac{\Delta x^2}{12}u_{xxxx} + O(\Delta x^4)$$

(b) Suppose that $u(x,t)$ is a smooth solution of the heat equation

$$u_t = u_{xx}$$

for $x \in (0,1)$ and $t > 0$. Construct the BTCS scheme for this equation (i.e. use a backward difference in time and a central difference in space at the point (x_j, t_{n+1})). Use the results from part (a) to show that the LTE for this method has the form

$$\text{LTE} = (a\Delta t + b\Delta x^2)u_{xxxx} + O(\Delta t^2) + O(\Delta x^4),$$

and calculate the constants a and b.

Show that the BTCS scheme can be written in the form

$$-r(w_{j-1}^{n+1} + w_{j+1}^{n+1}) + (1+2r)w_j^{n+1} = w_j^n, \quad j = 1, \ldots, J-1$$

where $n \geq 0$. Show by considering solutions of the form $w_j^n = \xi^n e^{i\omega j}$, that the scheme is unconditionally stable for all $\Delta t > 0$.

(c) Using the initial and boundary conditions specified, take $J = 4$, $r = 0.5$, and write down the equations for the BTCS scheme for the first time step. Evaluate numerically all the coefficients of the unknowns and the r.h.s., but do *not* attempt to solve these equations.

Question 2.

The reaction-diffusion equation $u_t = u_{xx} - u$ with $x \in [0, 1]$ is to be approximated on a uniform grid with mesh spacing $\Delta x = 1/J$. The boundary conditions are $u(0, t) = 1$ and $u(1, t) = 1$ and the initial conditions are $u(x, 0) = 1 + \sin(\pi x)$. The time step is Δt and the mesh ratio is $r = \Delta t/\Delta x^2$. The approximate solution is $w_j^n \approx u(j\Delta x, n\Delta t)$.

(a) Construct both the FTCS and the BTCS schemes for the reaction-diffusion equation above by approximating the partial derivatives with the appropriate difference operators. Construct the Crank-Nicholson (CN) scheme (θ-method with $\theta = \frac{1}{2}$) by averaging the FTCS and the BTCS schemes.

(b) Apply the CN scheme to the reaction-diffusion equation for one time step with $J = 2$, $r = 1$.

(c) Analyse the stability of the CN scheme for the reaction-diffusion equation using the von Neumann (Fourier) method.

Question 3.

(a) Construct the Upwind (FTBS) scheme for the linear hyberbolic equation $u_t + au_x = 0$ with $a > 0$. Perform a LTE analysis of the Upwind method to show that the method is in general first order in time and space.

(b) The Lax-Wendroff scheme for $u_t + au_x = 0$ is

$$w_j^{n+1} = (1-p^2)w_j^n - \frac{1}{2}p(1-p)w_{j+1}^n + \frac{1}{2}p(1+p)w_{j-1}^n,$$

where $p = a\Delta t/\Delta x$. By writing $w_j^n = \xi^n \exp(i\omega j)$, show that the scheme is von-Neumann stable for all $|p| \leq 1$.

(c) Discuss briefly the advantages and disadvantages of the Upwind scheme in comparison with the L-W scheme.

Question 4.

The 1D problem given by

$$\frac{d^2}{dx^2}u(x) = u(x) + f(x), \qquad u(0) = u(1) = 0,$$

is to be solved by minimising the functional

$$J[u] = \frac{1}{2}\int_0^1 \left\{ \left(\frac{du}{dx}\right)^2 + u^2 + 2f(x)u \right\} dx.$$

Show that the approximation $u(x) \approx w(x) = \sum_{k=1,\dots,N} c_k\phi_k(x)$ leads to the equations

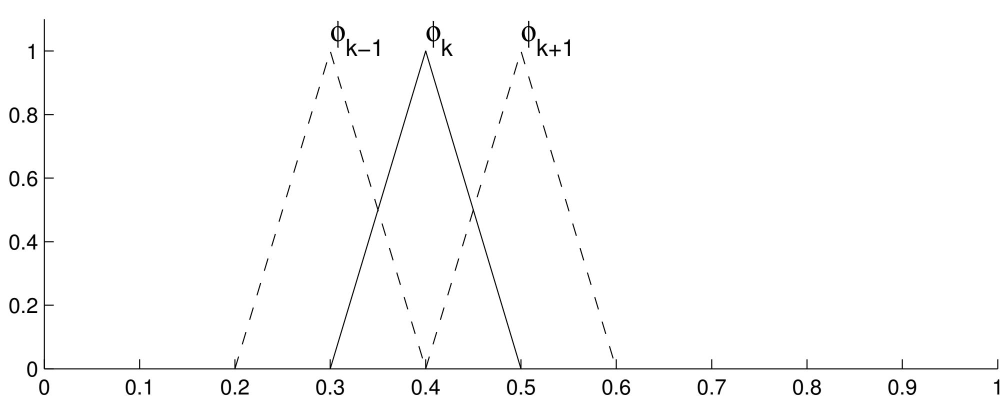
$$A\mathbf{c} = -\mathbf{b},$$

$$\text{where} \qquad \{a_{j,k}\} = \int_0^1 \left\{\phi_j'\phi_k' + \phi_j\phi_k\right\} dx,$$

$$\phi_k' = \frac{d\phi_k(x)}{dx}, \quad \text{and} \quad b_k = \int_0^1 f(x)\phi_k\, dx.$$

Let the ϕ_k be associated with the node points x_k chosen to be the piecewise linear function shown in the figure below, with $x_k - x_{k-1} = \Delta x$ for all k. Take $f(x) \equiv 1$. Hence, show that $a_{k,k} = 2/\Delta x + 2\Delta x/3$ and calculate $a_{k-1,k}$ and b_k.

[**Hint:** you may find it useful in evaluating some of the integrals to write $x_{k\pm1} = x_k \pm \Delta x$.]



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