Numerical Methods for PDEs

Partial Differential Equations

(Lecture 1, Week 1)

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Edinburgh, 12 January, 2015

Outline

1. Introduction
2. Classification of PDEs
3. Available numerical methods

Introduction and basic concepts

This course consists of the following four Sections:

1. *Partial Differential Equations* and the *Finite Difference Method*
2. *Parabolic PDEs*
3. *Hyperblic PDEs*
4. *Elliptic PDEs*

What is a partial differential equation?

Definition. *Equations which contain the partial derivatives of a function* $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ *are called Partial Differential Equations (PDEs):*

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x \partial y}\right) = 0\,.$$

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Examples of PDEs:

i) 1st order PDE:

$$\frac{\partial u}{\partial x} = u_x = 0\,.$$

$\Rightarrow$ Solutions $u(x, y)$ are invariant in x, hence $u(x, y) = \varphi(y)$.

ii) Linear transport or advection:

$$\begin{cases} u_t + cu_x = 0\,, & x \in \mathbb{R},\ t > 0 \\ u(0, x) = u_0(x)\,, & \text{“initial condition”} \end{cases}$$

$\Rightarrow$ Solution $u(t, x) = u_0(x - ct)$, since

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} = -cu_0'(x - ct) = -c\frac{\partial u}{\partial x}\,.$$

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Examples of PDEs (continued):

iii) Laplace equation: Let $\Omega \subset \mathbb{R}^2$. Find the solution of

$$\Delta u(x, y) := \mathrm{div}\,(\nabla u) := u_{xx} + u_{yy} = 0\,,$$

which requires boundary conditions for *uniqueness*. Possible solutions are

$$u(x, y) = x^2 - y^2\,,$$
$$u(x, y) = \ln\sqrt{x^2 + y^2}\,.$$

iv) Wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0\,, & x \in \mathbb{R},\ t > 0 \\ u(0, x) = A(x) \quad u_t(0, x) = B(x)\,, & \text{"initial conditions"} \end{cases}$$

$\Rightarrow$ Solution given by *d'Alembert's formula*

$$u(t, x) = \frac{1}{2}\left(A(x + ct) + A(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} B(\xi)\, d\xi\,.$$

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Examples of PDEs (continued)

v) Diffusion equation:

$$u_t - D\Delta u = 0\,,$$

where $D > 0$ is the diffusion constant.

v) Black-Scholes equation:

$$v_t + rsv_s + \frac{1}{2}\sigma^2 s^2 v_{ss} = rv\,,$$

where $v(s,t)$ is the value of a share option, s is the share price, r is the interest rate, and σ is the share "volatility".

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Classification of PDEs

Definition. *A linear PDE of the form*

$$a(x,y)u_{xx} + 2b(x,y)u_{x,y} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g\,,$$

is called

i) *elliptic in* $(x,y) \in \Omega$, *if* $ac - b^2 > 0$,

ii) *hyperbolic in* $(x,y) \in \Omega$, *if* $ac - b^2 < 0$,

iii) *parabolic in* $(x,y) \in \Omega$, *if* $ac - b^2 = 0$.

The above linear PDE is elliptic (hyperbolic, parabolic) if it is elliptic (hyperbolic, parabolic) for all $(x,y) \in \Omega$.

(These definitions can be generalized to higher number of dimensions and other orders)

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Examples:

i) Laplace equation: Let $\Omega \subset \mathbb{R}^2$. The equation

$$\Delta u(x,y) = u_{xx} + u_{yy} = 0\,,$$

is elliptic, since $a = c = 1$, $b = 0 \quad \Rightarrow \quad ac - b^2 = 1$.

ii) Wave equation: The equation

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is hyperbolic, since $a = 1$, $c = -1$, $b = 0 \quad \Rightarrow \quad ac - b^2 = -1$

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Examples (continued)

iii) The **diffusion equation**

$$u_t - D\Delta u = 0\,,$$

and the **Black-Scholes equation**

$$v_t + rsv_s + \frac{1}{2}\sigma^2 s^2 v_{ss} = rv\,,$$

are parabolic, since $a = 1$, $b = c = d = 0$, $e = -1$ (for diffusion) and $a = 1$, $b = c = 0$, $d = e = 1$ (for Black-Scholes) $\Rightarrow$ $ac - b^2 = 0$.

Numerical Methods for PDEs

Why?

- Often, no exact analytical solutions available
- Provide a systematic approximation of exact solutions (e.g error quantificiation)

Main numerical methods: (Advantages/Disadvantages)

1. **Finite Diference (FD) Methods.** Find discrete solutions on a (often rectangular) grid/mesh.
   (Simple / Complicated Domains, Discretised classical solutions)
2. **Finite Element (FE) Methods.** A class of Galerkin methods which are based on a partition of the domain into small finite elements.
   (Better in irregular domains / More complex to set up and analyze)
3. **Spectral Methods.** Solutions are approximated by a truncated expansion in the eigenfunctions of some linear operator (e.g. a truncated Fourier Series).
   (Highly accurate for problems with smooth solutions/ Not so useful on irregular domains or for problems with discontinuities)

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Summary of learning targets:

1. What is a PDE?
2. What types of PDEs exist and how are they classified?
3. What kind of numerical methods can be used? Advantages and disadvantages between them?