Numerical Methods for PDEs

Finite differences in higher spatial dimensions

(Lecture 10, Week 4)

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Edinburgh, February 2, 2013

Outline

1. FTCS scheme for the 2D heat equation
2. The 2D θ-method

The heat equation in 2D

The heat equation looks in 2D as follows

$$u_t = u_{xx} + u_{yy} \, ,$$

with solutions $u(x, y, t)$ and approximate solutions $w^n_{j,l} \approx u(x_j, y_l, t_n)$.

Now the spatial grid is 2D.

Applying the same ideas as in the 1D case, i.e., we approximate u_{yy} as u_{xx} by δ_y^2

$$\begin{aligned} \frac{w^{n+1}_{j,l} - w^n_{j,l}}{k} &= \frac{F_t}{k} w^n_{j,l} = \left(\frac{\delta_x^2}{h_x^2} + \frac{\delta_y^2}{h_y^2} \right) w^n_{j,l} \\ &= \frac{w^n_{j-1,l} - 2w^n_{j,l} + w^n_{j+1,l}}{h_x^2} + \frac{w^n_{j,l-1} - 2w^n_{j,l} + w^n_{j,l+1}}{h_y^2} \end{aligned}$$

giving the scheme

$$w^{n+1}_{j,l} = w^n_{j,l} + r_x(w^n_{j-1,l} - 2w^n_{j,l} + w^n_{j+1,l}) + r_y(w^n_{j,l-1} - 2w^n_{j,l} + w^n_{j,l+1})$$

with $r_x = k/h_x^2$, $r_y = k/h_y^2$.

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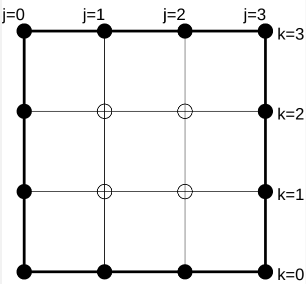
with $r_x = k/h^2_x$, $r_y = k/h^2_y$.

Example: the FTCS scheme for the 2D heat equation

Compute with the FTCS scheme two time steps of the problem

$$\begin{cases} u_t = u_{xx} + u_{yy} & \\ u(x,y,0) = \sin(\pi x/2)\sin(\pi y) & \text{ICs}\,, \\ u(0,y,t) = u(x,1,t) = u(x,0,t) = 0,\ \ u(1,y,t) = \sin(\pi y) & \text{BCs}\,, \end{cases}$$

on the unit square $[0,1]$ with $h_x = h_y = 1/3$ and $r = 0.25$.
The grid for this scheme looks like this



Initial conditions:

$$w^0_{j,l} = \sin(\pi j/6)\sin(\pi l/3), j = 0,\ldots,3; l = 0,\ldots,3$$

Boundary conditions:

$$w_{0,j} = w_{j,0} = w_{j,3} = 0, w_{3,j} = \sin(\pi j/3)$$

such that $w^0_{j,l}$ admits the values

$l \setminus j$	0	1	2	3
0	0	0	0	0
1	0	0.4330	0.7500	0.8660
2	0	0.4330	0.7500	0.8660
3	0	0	0	0

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First time step: Apply the FTCS scheme with $j = 1, 2;\ l = 1, 2,\ n = 0$ to get the values of $w^1_{j,l}$

$l \setminus j$	0	1	2	3
0	0	0	0	0
1	0	0.2958	0.5123	0.8660
2	0	0.2958	0.5123	0.8660
3	0	0	0	0

2nd time step: Apply the FTCS scheme with $j = 1, 2;\ l = 1, 2;\ n = 1$ to get the values of $w^2_{j,l}$

$l \setminus j$	0	1	2	3
0	0	0	0	0
1	0	0.2020	0.4185	0.8660
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2nd time step: Apply the FTCS scheme with $j = 1, 2;\ l = 1, 2;\ n = 1$ to get the values of $w_{j,l}^2$

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Note: The solution is symmetric about the line $y = 1/2$ as the ICs and BCs satisfy also this symmetry.

Stability of the FTCS scheme for the 2D heat equation

Main steps: The von Neumann stability method

1. Substitute $w_{j,l}^n = \xi^n \exp(ij\alpha) \exp(il\beta)$ into the FTCS scheme.
2. Divide through by $\xi^n \exp(ij\alpha) \exp(il\beta)$ and rearrange to get an expression for ξ.
3. Find conditions on the mesh ratios that guarantee $|\xi| \leq 1$ for all $(\alpha, \beta) \in [-\pi, \pi]^2$.

Step 1:

$$\begin{aligned}(\xi^{n+1} - \xi^n)\, e^{ij\alpha}\, e^{il\beta} &= r_x\, \xi^n\, e^{il\beta} \left[e^{i(j-1)\alpha} - 2e^{ij\alpha} + e^{i(j+1)\alpha} \right] \\ &\quad + r_y\, \xi^n\, e^{ij\alpha} \left[e^{i(l-1)\beta} - 2e^{il\beta} + e^{i(l+1)\beta} \right]\end{aligned}$$

Step 2:

$$\begin{aligned}\xi - 1 &= r_x \left(e^{-i\alpha} - 2 + e^{i\alpha} \right) + r_y \left(e^{-i\beta} - 2 + e^{i\beta} \right) \\ &= -4r_x \sin^2 \frac{\alpha}{2} - 4r_y \sin^2 \frac{\beta}{2}.\end{aligned}$$

Note: $(2\cos\alpha - 2) = -4\sin^2\frac{\alpha}{2}$ and $(2\cos\beta - 2) = -4\sin^2\frac{\beta}{2}$

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Step 3: *i) On a square spatial mesh ($h_x = h_y$):* We get

$$\xi = 1 - 4r\sin^2\frac{\alpha}{2} - 4r\sin^2\frac{\beta}{2} \quad \text{where } r = r_x = r_y.$$

We need to guarantee that $|\xi| \leq 1$. The max and min values of ξ occur at the maximum and minimum values **of the sine functions** (because $r \geq 0$), i.e. at $(\alpha, \beta) = (\pm\pi, \pm\pi)$ and $(\alpha, \beta) = (0, 0)$ respectively.

So

$$-1 \leq 1 - 8r \leq \xi \leq 1 \quad \text{for all } \alpha, \beta \in [-\pi, \pi].$$

For stability we therefore require $1 - 8r \geq -1$, i.e. the scheme is only stable when $r \leq 1/4$.

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ii) On a general spatial mesh with $h_x \neq h_y$: We have

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so that the scheme is stable if and only if

$$\begin{aligned} -1 \leq 1 - 4r_x - 4r_y &\leq 1, \\ 1 \geq 4(r_x + r_y) - 1 &\geq -1, \\ 2 \geq 4(r_x + r_y) &\geq 0, \\ 0 \leq r_x + r_y &\leq \frac{1}{2}. \end{aligned}$$

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Remark: *In the higher dimensional case it is even more important than before to develop schemes which are more efficient than the FTCS scheme:*

i) *At each time level, there is much more work (i.e. M^2 equations to calculate).*

ii) *When $r_x = r_y$, $h_x = h_y$, then the stability is twice as bad as in the 1D case.*

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The 2D θ-method

For simplicity we work with $\theta = \frac{1}{2}$, but the same principles apply for general values of $\theta > 0$. Write

$$\begin{aligned}
\frac{w_{j,l}^{n+1} - w_{j,l}^{n}}{k} &= \frac{F_t}{k} w_{j,l}^n = \frac{1}{2}\frac{\delta_x^2}{h_x^2}\left(w_{j,l}^n + w_{j,l}^{n+1}\right) + \frac{1}{2}\frac{\delta_y^2}{h_y^2}\left(w_{j,l}^n + w_{j,l}^{n+1}\right) \\
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\end{aligned}$$

and taking the unknown $w_{j,l}^{n+1}$ to the left gives

$$w_{j,l}^{n+1} - \frac{1}{2} r_x \delta_x^2 w_{j,l}^{n+1} - \frac{1}{2} r_y \delta_y^2 w_{j,l}^{n+1} = w_{j,l}^n + \frac{1}{2} r_x \delta_x^2 w_{j,l}^n + \frac{1}{2} r_y \delta_y^2 w_{j,l}^n .$$

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Disadvantages of the 2D θ-method

- Scheme shows five unknowns
- At each interior point (x_j, y_l), there are $(J-1)\times(L-1)$ equations for the unknowns $w_{j,l}^{n+1}$, $j = 1, \ldots, J-1; l = 1, \ldots, L-1$.
- The (sparse) matrix does not have a simple tri-diagonal structure anymore.

Remark: A system of N equations generally requires $\frac{1}{3}N^3$ floating point operations to solve it using Gaussian elimination. Hence, the above 2D scheme will require approximately $\frac{1}{3}J^3L^3$ operations.

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- At each interior point (x_j, y_l), there are $(J-1) \times (L-1)$ equations for the unknowns $w_{j,l}^{n+1}$, $j = 1, \ldots, J-1; l = 1, \ldots, L-1$.
- The (sparse) matrix does not have a simple tri-diagonal structure anymore.

Remark: A system of N equations generally requires $\frac{1}{3}N^3$ floating point operations to solve it using Gaussian elimination. Hence, the above 2D scheme will require approximately $\frac{1}{3}J^3L^3$ operations.

$\Rightarrow$ Already small $J = L = 10$ leads to 3×10^5 operations at each time step.