Numerical Methods for PDEs

Alternating direction implicit (ADI) schemes

(Lecture 11, Week 4)

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Outline

1. The ADI method

A more practical 2D scheme: The ADI method

For u smooth enough, the Taylor expansion of $u(x_j + h_x, y_l, t_n)$ admits an **Exponential operator notation**

$$
\begin{aligned}
u(x_j+h_x, y_l, t_n) &= \left[u + h_x \frac{\partial u}{\partial x} + \frac{h_x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h_x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \ldots\right]_{(x_j,y_l,t_n)} \\
&= \left[1 + h_x \frac{\partial}{\partial x} + \frac{h_x^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{h_x^3}{3!}\frac{\partial^3}{\partial x^3} + \ldots\right] u \,\big|_{(x_j,y_l,t_n)} \\
&= \exp\left(h_x \frac{\partial}{\partial x}\right) u \,\big|_{(x_j,y_l,t_n)}.
\end{aligned}
$$

Similarly (missing out the steps in between) we can write

$$
u(x_j, y_l + h_y, t_n) = \exp\left(h_y \frac{\partial}{\partial y}\right) u \,\big|_{(x_j,y_l,t_n)}
$$

and

$$
u(x_j, y_l, t_n + k) = \exp\left(k \frac{\partial}{\partial t}\right) u \,\big|_{(x_j,y_l,t_n)}
$$

With the help of this notation, we derive a different implicit scheme for

$$u_t = u_{xx} + u_{yy} \,.$$

Suppose that $u(x, y, t)$ is a smooth solution of the PDE. Taylor expanding $u(x, y, t_n + k)$ about (x, y, t_n) gives

$$u(x, y, t_n{+}k) = \exp\left(k\frac{\partial}{\partial t}\right) u\,\big|_{(x,y,t_n)} \,.$$

Use $e^a = e^{a/2} \cdot e^{a/2}$ to rewrite this as

$$u|_{t=t_n+k} = \exp\left(\frac{k}{2}\frac{\partial}{\partial t}\right) \exp\left(\frac{k}{2}\frac{\partial}{\partial t}\right) u|_{t=t_n}$$

and hence

$$\underbrace{\exp\left(-\frac{k}{2}\frac{\partial}{\partial t}\right) u|_{t=t_n+k}}_{\textbf{new time-level}} = \underbrace{\exp\left(\frac{k}{2}\frac{\partial}{\partial t}\right) u|_{t=t_n}}_{\textbf{old time-level}} \,. \qquad [*]$$

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We now use the fact that u solves the PDE to write

$$\begin{aligned}\exp\left(\pm\frac{k}{2}\frac{\partial}{\partial t}\right)u &= \exp\left(\pm\frac{k}{2}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\right)u \\ &= \exp\left(\pm\frac{k}{2}\frac{\partial^2}{\partial x^2}\right)\exp\left(\pm\frac{k}{2}\frac{\partial^2}{\partial y^2}\right)u\end{aligned}$$

(using $e^{a+b} = e^a \cdot e^b$). Plugging this into [$*$] gives

$$\begin{aligned}&\exp\left(-\frac{k}{2}\frac{\partial^2}{\partial x^2}\right)\exp\left(-\frac{k}{2}\frac{\partial^2}{\partial y^2}\right)u\big|_{t=t_n+k} \\ &= \exp\left(\frac{k}{2}\frac{\partial^2}{\partial x^2}\right)\exp\left(\frac{k}{2}\frac{\partial^2}{\partial y^2}\right)u\big|_{t=t_n}.\end{aligned}$$

We now chop the exponentials at first order ($e^{\pm q} \approx 1 \pm q$) to get

$$\left(1-\frac{k}{2}\frac{\partial^2}{\partial x^2}\right)\left(1-\frac{k}{2}\frac{\partial^2}{\partial y^2}\right)u\big|_{t=t_n+k} \approx \left(1+\frac{k}{2}\frac{\partial^2}{\partial x^2}\right)\left(1+\frac{k}{2}\frac{\partial^2}{\partial y^2}\right)u\big|_{t=t_n},$$

and with second central differences in space leads to

$$\left(1-\frac{r_x}{2}\delta_x^2\right)\left(1-\frac{r_y}{2}\delta_y^2\right)w_{j,l}^{n+1} = \left(1+\frac{r_x}{2}\delta_x^2\right)\left(1+\frac{r_y}{2}\delta_y^2\right)w_{j,l}^{n},$$

where r_x, r_y defined as before.

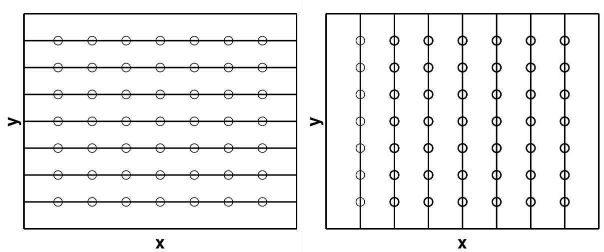
The scheme

$$\left(1-\frac{r_x}{2}\delta_x^2\right)\left(1-\frac{r_y}{2}\delta_y^2\right)w_{j,l}^{n+1}=\left(1+\frac{r_x}{2}\delta_x^2\right)\left(1+\frac{r_y}{2}\delta_y^2\right)w_{j,l}^{n},$$

it is **much easier and quicker to use than it looks** as it splits into two stages with an intermediate quantity $v_{j,l} := \left(1-\frac{r_y}{2}\delta_y^2\right)w_{j,l}^{n+1}$:

$$\left.\begin{array}{llcl} \textbf{Stage 1:} & \left(1-\frac{r_x}{2}\delta_x^2\right)v_{j,l} & = & \left(1+\frac{r_y}{2}\delta_y^2\right)w_{j,l}^{n} \\ \textbf{Stage 2:} & \left(1-\frac{r_y}{2}\delta_y^2\right)w_{j,l}^{n+1} & = & \left(1+\frac{r_x}{2}\delta_x^2\right)v_{j,l} \end{array}\right\} \textbf{ADI scheme}$$

The name **ADI** comes from this idea of alternately solving along the $x-$direction and $y-$direction.



The two step splitting solves the full scheme: Applying $\left(1 - \frac{r_x}{2}\delta_x^2\right)$ to Stage 2 gives

$$\begin{aligned} \left(1 - \frac{r_x}{2}\delta_x^2\right)\left(1 - \frac{r_y}{2}\delta_y^2\right) w_{j,l}^{n+1} &= \left(1 - \frac{r_x}{2}\delta_x^2\right)\left(1 + \frac{r_x}{2}\delta_x^2\right) v_{j,l} \\ &= \left(1 + \frac{r_x}{2}\delta_x^2\right)\left(1 - \frac{r_x}{2}\delta_x^2\right) v_{j,l} \quad \text{(difference operators commute (Why?))} \\ &= \left(1 + \frac{r_x}{2}\delta_x^2\right)\left(1 + \frac{r_y}{2}\delta_y^2\right) w_{j,l}^{n} \quad \text{by Stage 1.} \end{aligned}$$

Advantages:

Faster than 2D θ-method: Matrices are tridiagonal and involve only J or L unknowns with operations of $O(JL)$ compared to $O(J^3L^3)$ (for the θ-scheme).

Exercise for the remaining part of today's lecture:
Show that the 2D ADI scheme is *unconditionally stable*.

The **generalisation to 3D** (with 2 intermediate variables) of the 2D ADI **is not** unconditionally stable!

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&= \left(1 + \frac{r_x}{2}\delta_x^2\right)\left(1 - \frac{r_x}{2}\delta_x^2\right) v_{j,l} \quad \text{(difference operators commute (Why?))} \\
&= \left(1 + \frac{r_x}{2}\delta_x^2\right)\left(1 + \frac{r_y}{2}\delta_y^2\right) w_{j,l}^{n} \quad \text{by Stage 1.}
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