Numerical Methods for PDEs

Stability of the ADI method and an example of a 1D nonlinear problem

(Lecture 12, Week 4)

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Outline

1. Stability of the ADI scheme

2. LTE and stability of a nonlinear problem

Von Neumann stability of the ADI method

Ansatz: Substitute $w_{j,l}^n := \xi^n e^{i\omega_x j} e^{i\omega_y l}$ and $v_{j,l} := \overline{v}\xi^n e^{i\omega_x j} e^{i\omega_y l}$ into the ADI scheme and apply the short hand notations

$$X = 2r_x \sin^2(\omega_x/2) \qquad Y = 2r_y \sin^2(\omega_y/2)\,.$$

This leads to

$$\begin{cases} \overline{v} = (1-Y)/(1+X)\,, \\ \xi = \overline{v}(1-X)/(1+Y) = (1-X)/(1+X)\cdot(1-Y)/(1+Y)\,, \end{cases}$$

where we used that (Exercise:)

$$\delta_x^2\left[e^{i\omega_x j} e^{i\omega_y l}\right] = -4\sin^2(\omega_x/2)\left[e^{i\omega_x j} e^{i\omega_y l}\right]$$
$$\delta_y^2\left[e^{i\omega_x j} e^{i\omega_y l}\right] = -4\sin^2(\omega_y/2)\left[e^{i\omega_x j} e^{i\omega_y l}\right]$$

Hence, $|\xi| \leq 1$ for all harmonics ω_x, ω_y since

$$|(1-X)/(1+X)| \leq 1, \text{ and } |(1-Y)/(1+Y)| \leq 1 \qquad \forall X,\ Y \geq 0\,.$$

The LTE and stability of a nonlinear parabolic equation

We consider the nonlinear equation

$$u_t - u_{xx} + a(u)u = 0 \qquad (*)$$

for $0 < a_1 < a(\cdot) < a_2$ and $a(\cdot)$ smooth.

Approximate the linear part by the BTCS method. Then, an immediate discretisation strategy for the nonlinear term is

$$\frac{w_j^n - w_j^{n-1}}{k} - \frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{h^2} + a(w_j^{n-1})w_j^n = 0\,. \qquad (**)$$

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Compute the LTE for $(**)$:

Step 1: Substitute $w_j^n \to u(x_j, t_n)$.

Step 2: Apply a Taylor expansion at time t_n since most of the terms in $(**)$ are at the time level n. For $u_j^n := u(x_j, t_n)$ we have

$$\frac{u_j^n - u_j^{n-1}}{k} = u_t + O(k),$$

and

$$\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} = u_{xx} + O(h^2).$$

The non-standard term $a(u_j^{n-1})u_j^n$ is expanded at u_j^n with $u^* = (1-\theta)u_j^{n-1} + \theta u_j^n$ for some $0 \le \theta \le 1$ by

$$\begin{aligned} a(u_j^{n-1}) &= a(u_j^n) - a'((1-\theta)u_j^{n-1} + \theta u_j^n)(u_j^n - u_j^{n-1}) \\ &= a(u_j^n) - a'(u^*)k\left(\frac{u_j^n - u_j^{n-1}}{k}\right) \\ &= a(u_j^n) - a'(u^*)k(u_t(x_j, t_n) + O(k)). \end{aligned}$$

Since $a(\cdot)$ is sufficiently smooth (i.e., $|a'(\cdot)| < C$), we get

$$a(u_j^{n-1}) = a(u_j^n) + O(k).$$

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Step 3: Collect all the results and use $(*)$ in

$$LTE = \mathrm{LOT}\left[L_{k,h}u(x_j,t_n)\right],$$

i.e.,

$$\begin{aligned} LTE &= \mathrm{LOT}\left[\frac{u_j^n-u_j^{n-1}}{k} - \frac{u_{j-1}^n-2u_j^n+u_{j+1}^n}{h^2} + a(u_j^{n-1})u_j^n\right] \\ &= \left[u_t + O(k) - u_{xx} + O(h^2) + (a(u)+O(k))u\right]_{x_j,t_n} \\ &= u_t - u_{xx} + a(u)u + O(k) + O(h^2) + O(k) \\ &= O(k) + O(h^2). \end{aligned}$$

Compute the von Neumann stability for $(**)$:

Step 1: Substitute $w_j^n = \xi^n e^{i\omega j}$ in $(**)$ and multiply by k, that is,

$$\xi^n e^{i\omega j} - \xi^{n-1}e^{i\omega j} + r(-\xi^n e^{i\omega(j-1)} + 2\xi^n e^{i\omega j} - \xi^n e^{i\omega(j-1)}) + k\,a(\xi^{n-1}e^{i\omega j})\xi^n e^{i\omega j} = 0.$$

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Then, divide by $\xi^{n-1} e^{i\omega j}$ and set $a(*) = a(\xi^{n-1} e^{i\omega j})$ to get

$$\xi - 1 + r(-\xi e^{-i\omega} + 2\xi - \xi e^{i\omega}) + k\, a(*)\xi = 0.$$

Using $e^{i\omega} + e^{-i\omega} - 2 = -4\sin^2(\omega/2)$ leads to

$$\xi(1 + r4\sin^2(\omega/2) + k\, a(*)) = 1,$$

and after rewriting

$$\xi = \frac{1}{(1 + r4\sin^2(\omega/2) + k\, a(*))}.$$

Since $0 < a_1 < a(*)$ the scheme is always stable.