Numerical Methods for PDEs

Introduction to Hyperbolic PDEs

(Lecture 13, Week 5)

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Outline

1. Examples

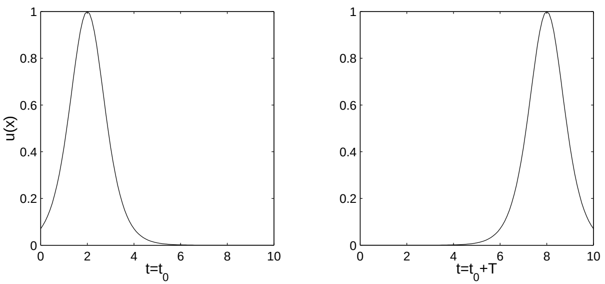
2. Simple numerical schemes for the advection equation

What kind of processes are described by Hyperbolic PDEs?

Wave propagation phenomena such as

waves in water, gas, plasmas, traffic flow, etc.

If there is no dissipation (loss of energy), then the wave keeps its form



The previous figure showed a wave

(1) moving from left to right with a certain speed

(2) with constant wave form

The simplest hyperbolic equation capturing **(1)** and **(2)** is the **first order advection equation**

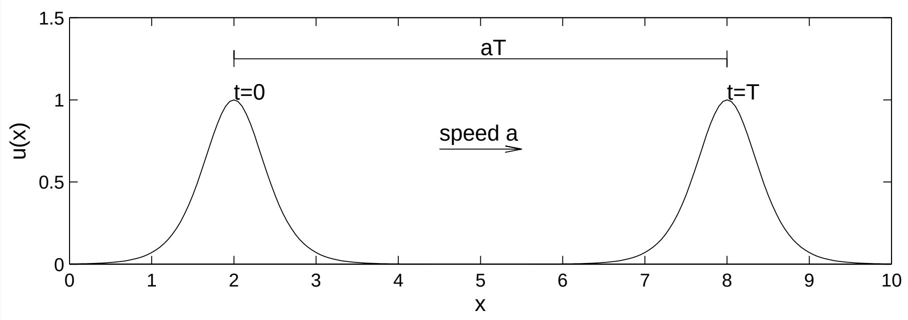
$$\textbf{(AE)} \qquad \begin{cases} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, & a = \text{const.}\,, \\ u(x,0) = F(x) & \text{initial condition}\,. \end{cases}$$

Equation **(AE)** is a useful test for numerical schemes approximating hyperbolic PDEs.

Remark: $u(x,t) = F(x-at)$ is an exact solution of **(AE)**, since $\frac{\partial u}{\partial t} = (-a)F'(x-at)$ and $\frac{\partial u}{\partial x} = F'(x-at)$.

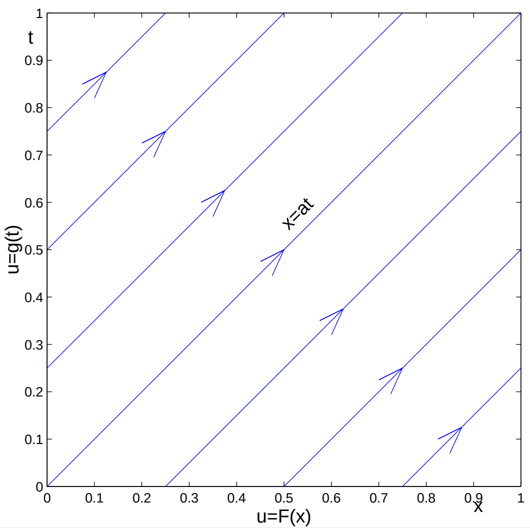
Propagation with constant speed

Meaning of the analytical solution: Initial wave form defined by F moves with constant speed a to the right if $a > 0$ and to the left if $a < 0$.



The solution is constant along each *characteristic line* with slope

$$dx/dt = a.$$



Other examples of Hyperbolic PDEs

- advection equation with variable coefficient

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0$$

- The wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

- higher dimensions

$$\frac{\partial u}{\partial t} + a_x\frac{\partial u}{\partial x} + a_y\frac{\partial u}{\partial y} = 0$$

- nonlinear equations, for example Burger's equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = 0$$

Boundary conditions for **(AE)**

Consider the equation **(AE)** on the interval $(0,1)$, then only one boundary condition is required, i.e.,

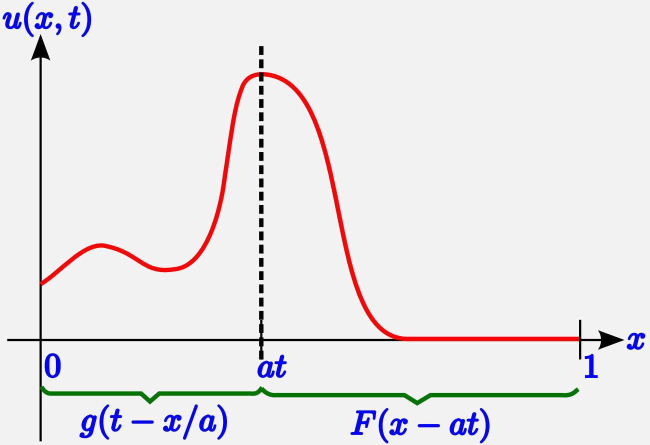
$$\text{if } \left\{ \begin{array}{l} a > 0 \\ a < 0 \end{array} \right\} \text{we specify } \left\{ \begin{array}{l} u(0,t) = g(t) \\ u(1,t) = g(t) \end{array} \right.$$

Example: Let $a > 0$ and

$$\begin{cases} u(x,0) = F(x) & \text{initial condition}\,, \\ u(0,t) = g(t) & \text{left-hand boundary condition}\,, \end{cases}$$

leads to the exact solution

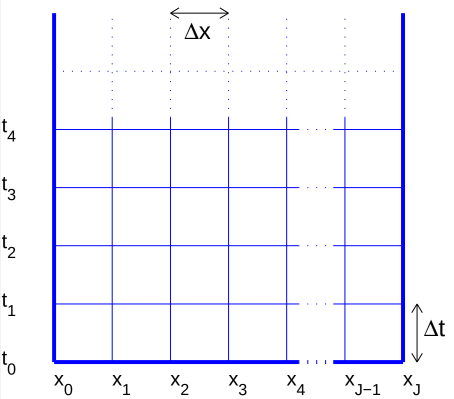
$$u(x,t) = \begin{cases} g(t - x/a) & x \le at\,, \\ F(x - at) & x > at\,. \end{cases}$$



F contains the information from the initial condition
g represents the new information induced by the left-hand BC

Simple discretisations of the advection equation

We use the usual uniform grid in x and t, i.e. fixed values of h and k.



with $x_j = x_0 + jh$, $t_n = nk$, and the approximate solution $u(x_j, t_n) \approx w_j^n$.

Simple explicit scheme

Advection Equation (AE):

$$u_t + au_x = 0$$

Forward difference approximation in time:

$$u_t \approx \frac{F_t}{k} w_j^n = \frac{w_j^{n+1} - w_j^n}{k}$$

Different approximations in space:

$$u_x\big|_{(x_j,t_n)} \approx \begin{cases} (w_j^n - w_{j-1}^n)/h, & \text{backwards diff.} \\ (w_{j+1}^n - w_{j-1}^n)/2h, & \text{central diff.} \\ (w_{j+1}^n - w_j^n)/h, & \text{forwards diff.} \end{cases}$$

FTBS and FTFS schemes

With the backward difference operator B_x we get the **FTBS scheme**

$$\frac{w_j^{n+1} - w_j^n}{k} + a\frac{w_j^n - w_{j-1}^n}{h} = 0$$

or

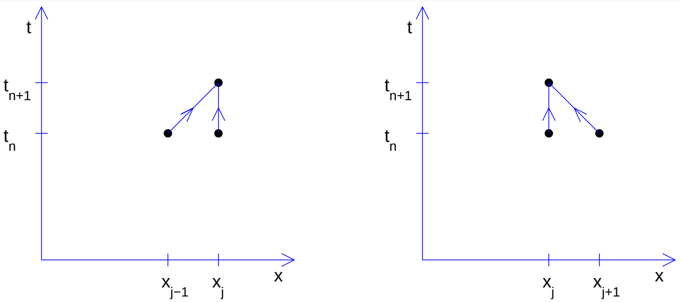
$$w_j^{n+1} = (1-p)w_j^n + pw_{j-1}^n$$

where

$$p = \frac{ak}{h} = \text{CFL number (Courant-Friedrichs-Lewy, 1928)}$$

Alternatively, with the forward difference operator F_x we get the **FTFS scheme**

$$w_j^{n+1} = (1+p)w_j^n - pw_{j+1}^n\,.$$



FTBS scheme: Info "travels" to the right (like PDE with $a > 0$).

FTFS scheme: Info "travels" to left (like PDE with $a < 0$).

Interpretation: Applicability of the schemes depends on $\text{sign}(a)$.

We will check this with the LTE and the stability of the schemes.