Numerical Methods for PDEs

Hyperbolic PDEs: LTE and Stability of FTBS and FTFS scheme; the FTCS scheme

(Lecture 14, Week 5)

Markus Schmuck

Department of Mathematics and Maxwell Institute for Mathematical Sciences
Heriot-Watt University, Edinburgh

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Outline

1. LTE and stability of the FTBS scheme
2. LTE and stability of the FTFS scheme
3. The FTCS scheme

LTE of the FTBS scheme

For the FTBS scheme we have

$$L_{h,k} w_j^n = \frac{w_j^{n+1} - w_j^n}{k} + a\frac{w_j^n - w_{j-1}^n}{h},$$

and the LTE is computed by

$$\begin{aligned}
\text{LTE} &:= \mathrm{LOT}\left[L_{h,k} u(x_j, t_n)\right] = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{k} + a\frac{u(x_j, t_n) - u(x_{j-1}, t_n)}{h} \\
&= \left(u_t + \frac{1}{2} k u_{tt} + O(k^2) + a\left[u_x - \frac{1}{2} h u_{xx} + O(h^2) \right] \right)\Big|_{(x_j, t_n)} \\
&= \underbrace{(u_t + a u_x)}_{=0 \text{ by PDE}} + \underbrace{\frac{1}{2}(k\, u_{tt} - a h\, u_{xx})}_{\neq 0 \text{ in general}} + O(k^2, h^2),
\end{aligned}$$

so the FTBS scheme is in general **1st order accurate in time and space.**

The FTBS scheme is exact for $p = 1$:

$$\text{Note: } u_t + au_x = 0 \Rightarrow u_{tt} = -au_{xt} = -a(u_t)_x = a^2 u_{xx}$$

$$\text{so} \quad \frac{1}{2}k\,u_{tt} - \frac{1}{2}ah\,u_{xx} = \frac{1}{2}a[ak - h]u_{xx}$$
$$= 0 \quad \text{iff } p = 1 \quad (p = ak/h).$$

Exercise: Show that also all the higher order terms vanish for $p = 1$.

Stability of the FTBS scheme

Step 1: Insert $w_j^n = \xi^n e^{i\omega j}$ into

$$w_j^{n+1} = (1-p)w_j^n + pw_{j-1}^n$$

Step 2: Cancel the terms $\xi^n e^{i\omega j}$, i.e.,

$$\xi = (1-p) + pe^{-i\omega}$$

Using $e^{-i\omega} = \cos\omega - i\sin\omega$ gives

$$\begin{aligned} \xi &= (1-p+p\cos\omega) + p(-i\sin\omega) \\ \text{so} \quad |\xi|^2 &= (1-p+p\cos\omega)^2 + p^2\sin^2\omega \\ &= (1-p)^2 + 2(1-p)p\cos\omega + p^2\cos^2\omega + p^2\sin^2\omega \\ &= 1 - 2p(1-p)(1-\cos\omega) \\ &= 1 - 4p(1-p)\sin^2(\omega/2) \end{aligned}$$

Step 3: Stability requires $|\xi| \leq 1$.

- If $a > 0$, we get

$$4p(1-p)\sin^2(\omega/2) \geq p(1-p) \geq 0 \qquad (*)$$

since $p := ak/h$. (*) is satisfied for $p \in [0,1]$ and hence for $k \leq h/a$.

- If $a < 0$, then the FTBS scheme is unstable for all k.

LTE and stability of the FTFS scheme

The same steps we can repeat with the FTFS scheme:

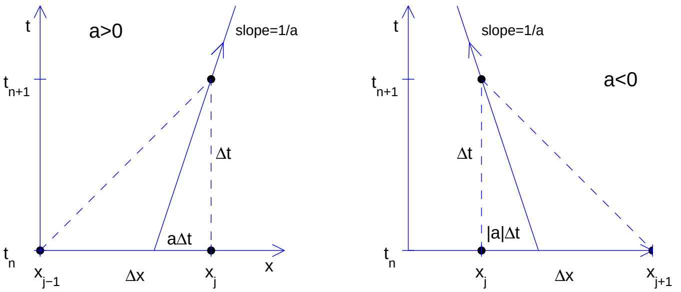
Exercise 1: Show that the FTFS scheme is also 1st order accurate, unless $p = -1$ ($h = -ak \Rightarrow a < 0$), in which case it is exact, i.e. $LTE = 0$.

Exercise 2: Show that the FTFS scheme is stable $\Leftrightarrow p \in [-1, 0]$, i.e. it is stable when $a < 0$ and $k \leq h/|a|$, and it is unstable if $a > 0$.

FTBS and FTFS scheme: Stability requirements

Visual interpretation of the stability requirement:

The characteristic line of the exact solution passing through (x_j, t_n) must lie within the “computational molecule”



FTBS scheme:
$h \geq ak$ for stability.

FTFS scheme:
$h \geq |a|k$ for stability.

The FTCS scheme

Expectations of using a central difference for u_x:

- higher order accuracy
- not capturing the “prefered direction” of hyperbolic problems

Resulting FTCS scheme:

$$w_j^{n+1} = w_j^n - \frac{1}{2}p(w_{j+1}^n - w_{j-1}^n).$$

Stability analysis:

$$\xi = 1 - \frac{1}{2}p(e^{i\omega} - e^{-i\omega}) = 1 - ip\sin(\omega),$$

and therefore

$$\begin{aligned} |\xi|^2 &= 1 + p^2\sin^2(\omega) \\ &> 1 \quad \text{for all } \omega \neq 0, \pm\pi \text{ for any } p \neq 0, \end{aligned}$$

Result: The FTCS scheme is *completely unstable* independent of a and p