Numerical Methods for PDEs

Hyperbolic PDEs: Relative phase error, test problems, and the Leapfrog scheme

(Lecture 15, Week 5)

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Outline

Relative phase errors: Complex amplification factors

Relative error δv: Let v be the true value and v_0 the measured/approximated value. Then

$$\delta v := \frac{\Delta v}{v} = \frac{v_0 - v}{v} = \frac{v_0}{v} - 1$$

1) Fourier/von Neumann stability:
Fourier mode $F(x) := e^{i\alpha x}$ as initial condition at $t = 0$

2) Exact solution of Advection Equation (AE) for IC given by 1):
Let $a > 0$ in (AE), then

$$u(x,t) = e^{i\alpha(x-at)}$$

3) $u(x,t)$ on the usual spatio-temporal grid ($h := \Delta x,\ k := \Delta t$):

$$u(x_j, t_n) = u_j^n = \exp\left(i\alpha(jh - ank)\right) = \exp\left(-i\alpha ank\right)\exp\left(i\alpha jh\right) = \xi^n e^{i\omega j}$$

where $\omega = \alpha h$ and $\xi = e^{-i\alpha ak}$

4) The "true" phase shift ϕ for correct wave speed: Rewriting $\xi = |\xi| e^{i\arg\xi} = |\xi| e^{i\phi}$ with

$$|\xi| = \left(\mathrm{Im}(\xi)^2 + \mathrm{Re}(\xi)^2\right)^{1/2} \quad \text{and} \quad \phi = \arctan\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)$$

induces $\phi \approx -\alpha ak = -\omega ak/h = -\omega p$ as $h, k \to 0$.

5) Approximate phase shift ϕ_0 of FTBS scheme:
Recall that $\xi = (1 - p + p\cos\omega) + i(-p\sin\omega)$

and with the expansions $\sin x = x - x^3/3! + \dots$, $\cos x = 1 - x^2/2! + \dots$, and $\tan^{-1} x = x - x^3/3 + \dots$ we approximate ϕ_0 up to $\mathcal{O}(\omega^2)$, i.e.,

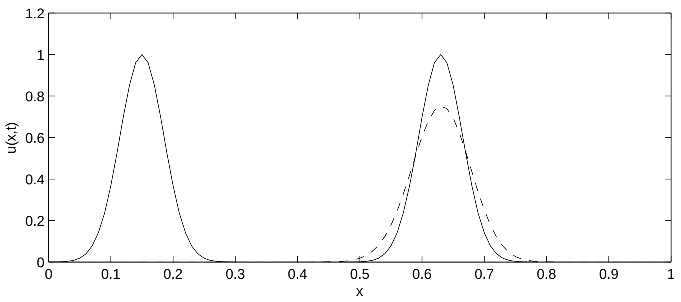
$$\begin{aligned}\phi_0 &= \tan^{-1}\left(\frac{-p\sin\omega}{1-p+p\cos\omega}\right)\\ &= -\tan^{-1}\left(\frac{p\sin\omega}{1-p+p\cos\omega}\right)\\ &\overset{(*)}{\approx} -p\omega\left[1-\frac{1}{6}(1-3p+2p^2)\omega^2+\dots\right],\end{aligned}$$

where we used the geometric series formula $1/(1-q)$ in $(*)$ for $q := -p\omega^2/2 = -ak\alpha h (= -\phi h > -1)$ and that the only term up to $\mathcal{O}(\omega^2)$ in $-1/3(p\omega)^2\left(1-\omega^2/6+p\omega^2/2\right)^3$ is $-p^2\omega^2/3$. Hence, the relative phase error becomes

$$\delta\phi = \frac{\phi_0-\phi}{\phi} = -\frac{1}{6}(1-3p+2p^2)\omega^2+\dots .$$

(1) First test problem: Gaussian pulse as initial condition

$$u(x,0) = \exp\left[-400(x-0.15)^2\right].$$



Initial "spike" $u(x,0)$ on the left. Exact solution $u(x,t)$ (solid line) of the PDE **(AE)** at $t = 0.5$ (r.h.spike, solid line). Approximate solution of FTBS scheme (dashed line) at $t = 0.5$.

(2) Second test problem: Step function as initial condition

$$u(x,0) = \begin{cases} 1 & \text{for } 0 \le x \le 0.15 \\ 0 & \text{for } 0.15 < x \le 1. \end{cases}$$

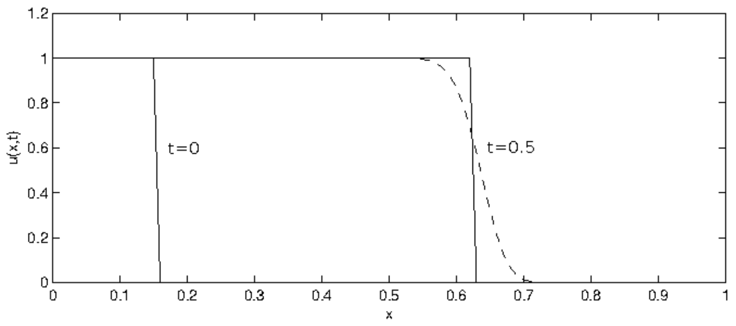


Figure shows the exact solution $u(x,t)$ (solid line) of **(AE)** at $t = 0.5$ and approximate solution (dashed line) of the FTBS scheme at $t = 0.5$.

Discussion of the test problems

Observation in problem (1) and (2):
Solutions lose amplitude and spread out, but their position is correct.

Recall the LTE of the FTBS scheme:

$$LTE = LOT\left[L_{h,k}u(x_j,t_n)\right] = \underbrace{u_t + au_x}_{\text{PDE}} + \underbrace{\frac{a}{2}(p-1)h\,u_{xx}}_{\text{LTE leading term}} + O(k^2,h^2)$$

Interpretation of LTE: The FTBS scheme approximates the equation

$$u_t + au_x - \alpha u_{xx} = 0\,,$$

where $\alpha = \frac{a}{2}(1-p)dx$. In this case the $O(k^2,h^2)$ would be its local truncation error.

Remark: The extra term is a *diffusion term* and historically this is called *artificial viscosity*.

The Leapfrog scheme for the PDE **(AE)**

The Leapfrog scheme, also called the CTCS scheme, reads

$$\frac{w_j^{n+1} - w_j^{n-1}}{2k} + a\frac{w_{j+1}^n - w_{j-1}^n}{2h} = 0$$
$$\Rightarrow w_j^{n+1} = w_j^{n-1} - p(w_{j+1}^n - w_{j-1}^n)$$

where $p := ak/h$.

Exercise: Show that the LTE of the leapfrog scheme is

$$\begin{aligned} \text{LTE} &= \frac{1}{6}\left[k^2 u_{ttt} + ah^2 u_{xxx}\right] + O(k^4, h^4) \\ &= \frac{a}{6}(1-p^2)u_{xxx}h^2 + O(k^4, h^4), \quad \text{using } u_{ttt} = -a^3 u_{xxx}, \end{aligned}$$

i.e. the Leapfrog scheme is *2nd order accurate*.