Numerical Methods for PDEs

Hyperbolic PDEs: The leapfrog scheme (LTE, stability & phase error) and the Lax-Wendroff scheme (LTE, stability & phase error)

(Lecture 16, Week 6)

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Outline

1. Stability of the leapfrog scheme
2. The phase shift of the leapfrog scheme
3. The Lax-Wendroff scheme
4. LTE, stability, and phase shift of the Lax-Wendroff scheme

Stability of the leapfrog scheme

Leapfrog scheme: (or CTCS method)

$$\frac{w_j^{n+1} - w_j^{n-1}}{2k} + a\frac{w_{j+1}^n - w_{j-1}^n}{2h} = 0$$
$$\Rightarrow w_j^{n+1} = w_j^{n-1} - p(w_{j+1}^n - w_{j-1}^n) \qquad (*)$$

Step 1: Insert the ansatz $w_j^n = \xi^n e^{i\omega j}$ into $(*)$, that is,

$$\xi^2 = 1 - p\xi\left(e^{i\omega} - e^{-i\omega}\right)$$
$$\text{or} \quad \xi^2 + 2ip\xi\sin\omega - 1 = 0.$$

For $a = 1,\ b = 2ip\sin\omega$ and $c = -1$, we obtain the roots

$$\xi_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -ip\sin\omega \pm \sqrt{1 - p^2\sin^2\omega}.$$

Step 2: Stability requires $|\xi_\pm| \leq 1$. The discriminant $1 - p^2 \sin\omega$ induces two cases:

- Case $|p| > 1$: Worst case for $\sin\omega = 1$, hence

$$\xi_\pm = -ip \pm i\sqrt{p^2 - 1} = -i\left[p \mp \sqrt{p^2 - 1}\right]$$

with $|\xi_-| > 1$. Hence, the scheme is unstable.

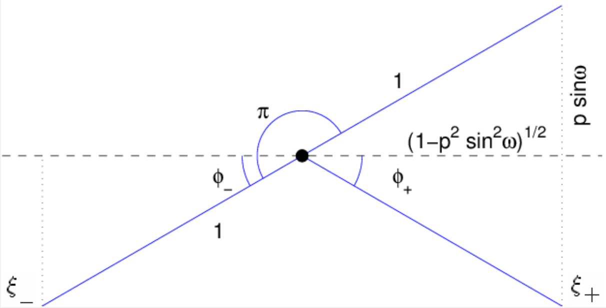
- Case $|p| \leq 1$: The discriminant is real for all ω and therefore

$$|\xi_\pm|^2 = (-p\sin\omega)^2 + (1 - p^2\sin^2\omega) = 1 \quad \forall\omega,$$

and hence $(*)$ is stable for all $|p| \leq 1$ (independent of $a > 0$ or $a < 0$).

Phase shift of the leapfrog method

$$\xi_\pm = -ip\sin\omega \pm \sqrt{1 - p^2\sin^2\omega}$$



Phase shift of ξ_+:

$$\xi_+ : \quad \phi_+ = -\sin^{-1}(p\sin\omega)$$
$$= -\sin^{-1}(p\omega - p\omega^3/3! + \dots)$$
$$= -p\omega(1 - \tfrac{1}{6}(1-p^2)\omega^2 + \dots)$$

(using $\sin^{-1}(x) = x + x^3/6 + \dots$)

$\Rightarrow$ Phase error of same sign $\forall n > 0$

Phase shift of ξ_-:

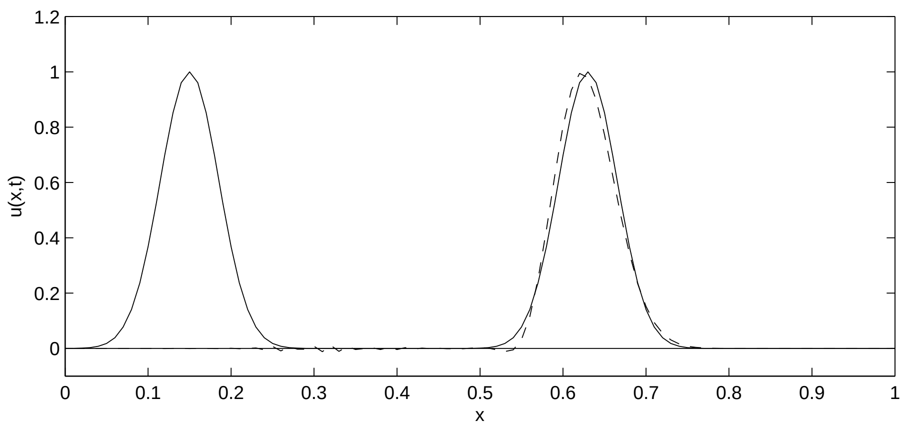
$$\phi_- = \pi + \sin^{-1}(p\sin\omega)$$
$$= p\omega + \pi - \tfrac{1}{6}p\omega^3(1-p^2) + \dots$$

$\Rightarrow$ Phase shift changes sign, since $w_j^n = \xi^n e^{i\omega j}$, hence $\phi_- n = n\pi + \dots$

$\Rightarrow$ Oscillations (require “filtering”)

Leapfrog scheme: Test example 1

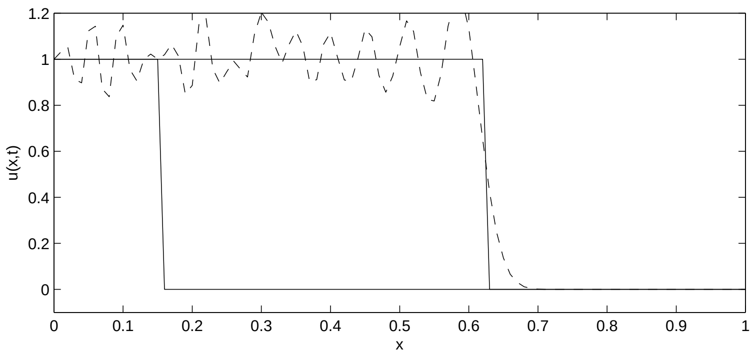
Initial condition: Gaussian pulse



Observation: “Only” small amplitude oscillations to the left

Leapfrog scheme: Test example 2

Initial condition: Step function



Observation: Step front better approximated than in the *upwind case* (i.e., FTBS for $a > 0$ and FTFS for $a < 0$) **but strong oscillations**

Recall: $LTE = Cu_{xxx} + O(k^4, h^4)$ is dispersive and not diffusive u_{xx}

The Lax-Wendroff scheme

Starting point: Leapfrog method shows strong oscillations/dispersion

Goal: Remove oscillations

Idea: Add a diffusive term by a Taylor expansion up to 2nd order, i.e., for u smooth solving **(AE)** use $\left(\frac{\partial}{\partial t}\right)^m u = \left(-a\frac{\partial}{\partial x}\right)^m u$, and

$$\begin{aligned} u(x, t+k) &= u + ku_t + \frac{1}{2}k^2 u_{tt} + O(k^3)\Big|_{(x,t)}, \\ &= u - aku_x + \frac{1}{2}a^2k^2u_{xx} + O(k^3)\Big|_{(x,t)}, \\ &\approx u - ak\frac{D_x}{2h}u + \frac{1}{2}a^2k^2\frac{\delta_x^2}{h^2}u, \end{aligned}$$

where we truncated the expansion (last line) and replaced derivatives by their central difference approximations.

Now, use the result from previous slide, that is,

$$u(x, t+k) \approx u - ak\frac{D_x}{2h}u + \frac{1}{2}a^2k^2\frac{\delta_x^2}{h^2}u$$

as our numerical scheme. That means,

$$w_j^{n+1} = \underbrace{w_j^n - \frac{p}{2}(w_{j+1}^n - w_{j-1}^n)}_{\text{FTCS scheme}} + \underbrace{\frac{p^2}{2}(w_{j+1}^n - 2w_j^n + w_{j-1}^n)}_{\text{extra term}}$$

which gives after rearranging the following **Lax-Wendroff scheme**

$$w_j^{n+1} = (1-p^2)w_j^n - \frac{1}{2}p(1-p)w_{j+1}^n + \frac{1}{2}p(1+p)w_{j-1}^n. \qquad \textbf{(LW)}$$

LTE of the Lax-Wendroff method

The Lax-Wendroff scheme (LW):

$$L_\Delta w_j^n = \frac{w_j^{n+1} - w_j^n}{k} + a\frac{D_x}{2h} w_j^n - \frac{1}{2}a^2 k \frac{\delta_x^2}{h^2} w_j^n,$$

Computing the LTE:

$$\begin{aligned}\text{LTE} &= L_\Delta u(x_j, t_n) = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{k} + a\frac{D_x}{2h} u(x_j, t_n) \\ &\quad - \frac{1}{2}a^2 k \frac{\delta_x^2}{h^2} u(x_j, t_n) \\ &= \underbrace{\frac{1}{2}k[u_{tt} - a^2 u_{xx}]}_{=0} + \frac{1}{6}k^2 u_{ttt} + \frac{a}{6}h^2 u_{xxx} + O(k^3, h^4, kh^2) \\ &= (1 - p^2)\frac{a}{6}h^2 u_{xxx} + O(h^3).\end{aligned}$$

Hence the method is 2nd order accurate.

Stability of the Lax-Wendroff scheme

After inserting $w_j^n = \xi^n e^{i\omega j}$ into **(LW)** and simplifying, we get

$$\begin{aligned}\xi &= (1-p^2) - \frac{1}{2}p(1-p)e^{i\omega} + \frac{1}{2}p(1+p)e^{-i\omega} \\ &= 1 + p^2(\cos\omega - 1) - ip\sin\omega \\ &= 1 - 2p^2\sin^2(\omega/2) - ip\sin\omega \\ &= 1 - 2p^2\sin^2(\omega/2) - 2ip\sin(\omega/2)\cos(\omega/2).\end{aligned}$$

Hence,

$$\begin{aligned}|\xi|^2 &= \left[1 - 2p^2s^2\right]^2 + 4p^2s^2c^2, \text{ where } s = \sin(\omega/2), c = \cos(\omega/2) \\ &= 1 + 4p^2s^2(c^2-1) + 4p^4s^4 \\ &= 1 - 4p^2(1-p^2)s^4\end{aligned}$$

$\Rightarrow$ $|\xi|^2 \le 1$ for all $|p| \le 1$ and $|\xi|^2 > 1$ for all $|p| > 1$

$\Rightarrow$ Scheme is stable for all $|p| \le 1$

Phase shift in the Lax-Wendroff method

We have

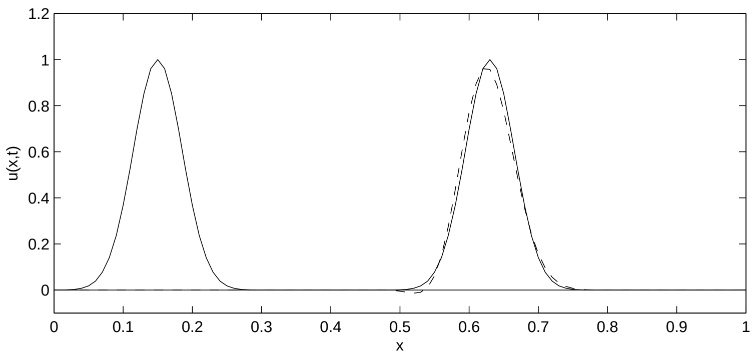
$$\begin{aligned}\phi &= -\tan^{-1}\left[\frac{p\sin\omega}{1-2p^2\sin^2(\omega/2)}\right]\\ &= -\tan^{-1}\left[p(\omega-\frac{1}{6}\omega^3+\dots)(1+p^2\omega^2/2+\dots)\right]\\ &= -\tan^{-1}\left[p\omega(1+\omega^2(\frac{1}{2}p^2-\frac{1}{6})+\dots)\right]\\ &= -p\omega\left(1-\frac{1}{6}\omega^2(1-p^2)+\dots\right).\end{aligned}$$

Observations:

- The same shift as for the first root of the Leapfrog scheme
- Second troublesome root of the Leapfrog scheme disappeared

Lax-Wendroff scheme: Test example 1

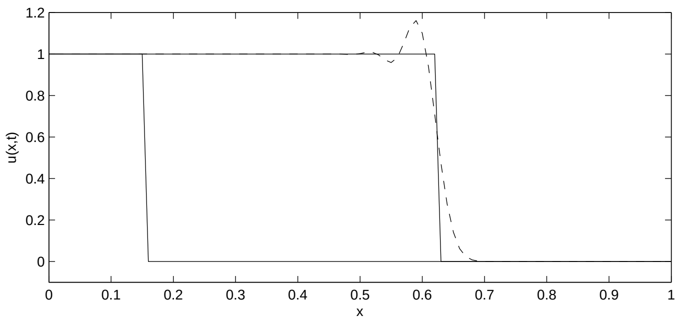
Initial condition: Gaussian pulse



Observation: The Gaussian is well-preserved by the scheme.

Lax-Wendroff scheme: Test example 2

Initial condition: Step function



Observations: • Less smeared out than the upwind schemes (i.e., FTFS for $a < 0$ & FTBS for $a > 0$), fewer oscillations than the leapfrog scheme (most are damped).

• The Lax-Wendroff is a well-used method.

• All the methods we have seen so far are explicit.