Numerical Methods for PDEs

Hyperbolic PDEs: Backward time schemes/The Crank-Nicolson scheme (LTE, stability & phase error)/Wave equation (LTE, stability & phase error)

(Lecture 17, Week 6)

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Outline

1. Backward time schemes
2. The Crank-Nicolson scheme (LTE, stability & phase error)
3. Wave equation (LTE, stability & phase error)

Backward time schemes: The BTCS scheme

The BTCS scheme is

$$\frac{w_j^{n+1} - w_j^n}{k} + a\frac{w_{j-1}^{n+1} - w_{j+1}^{n+1}}{2h} = 0$$

or

$$w_j^{n+1} + \frac{1}{2}p(w_{j+1}^{n+1} - w_{j-1}^{n+1}) = w_j^n$$

Exercise: Show that the BTCS scheme is first order in the LTE and that it is stable for all p.

The Crank-Nicolson (CN) scheme for **(AE)**

The PDE **(AE)** is $u_t = -au_x$. Approximate the spatial part by an average across time levels n and $n+1$, i.e.,

$$\frac{w_j^{n+1} - w_j^n}{k} \approx -\frac{a}{2}\left(u_x\,|_{t=t_n} + u_x|_{t=t_{n+1}}\right)$$
$$= -\frac{a}{2}\left(\frac{D_x}{2h}w_j^n + \frac{D_x}{2h}w_j^{n+1}\right),$$

using central differences in space to approximate u_x. If we rearrange all this we get the CN scheme for the advection equation.

$$w_j^{n+1} + \frac{p}{4}\left(w_{j+1}^{n+1} - w_{j-1}^{n+1}\right) = w_j^n - \frac{p}{4}\left(w_{j+1}^n - w_{j-1}^n\right)$$

As in the parabolic case, we need to solve a tridiagonal system of equations to get the solution at each timestep.

LTE, stability, and phase error of the CN method

LTE: Standard calculations (Exercise) show that

$$\text{LTE} = \underbrace{u_t + a u_x}_{=0} + \underbrace{\frac{k}{2}(u_{tt} + a u_{xt})}_{=0} + \frac{k^2}{6} u_{ttt} + \frac{ah^2}{6} u_{xxx} + \frac{ak^2}{4} u_{xtt} + \text{ h.o.t.}$$

$$= \frac{ah^2}{6} u_{xxx}\left(1 + \frac{1}{2}p^2\right) + O(h^3)$$

Stability: Insert $w_j^n = \xi^n e^{i\omega j}$ into the CN method

$$\xi \left[1 + \frac{p}{4}(e^{i\omega} - e^{-i\omega})\right] = 1 - \frac{p}{4}(e^{i\omega} - e^{-i\omega})$$

$$\Rightarrow \quad \xi = \frac{1 - \frac{1}{2} ip \sin\omega}{1 + \frac{1}{2} ip \sin\omega}$$

$$\Rightarrow \quad |\xi|^2 = \frac{|1 - \frac{1}{2} ip \sin\omega|^2}{|1 + \frac{1}{2} ip \sin\omega|^2} = \frac{1 + \frac{1}{4}p^2 \sin^2\omega}{1 + \frac{1}{4}p^2 \sin^2\omega} = 1 \quad \forall p, \omega$$

$\Rightarrow$ The CN scheme is stable *for all* p.

Phase error: We have

$$\phi = -\tan^{-1}\left[\frac{p\sin\omega}{1-\frac{p^2}{4}\sin^2(\omega)}\right]$$
$$= -\tan^{-1}\left[p\omega(1+\omega^2\left(\frac{p^2}{4}-\frac{1}{6}\right)+\ldots)\right]$$
$$= -p\omega\left(1-\frac{1}{6}\omega^2(1+\frac{1}{2}p^2)+\ldots\right).$$

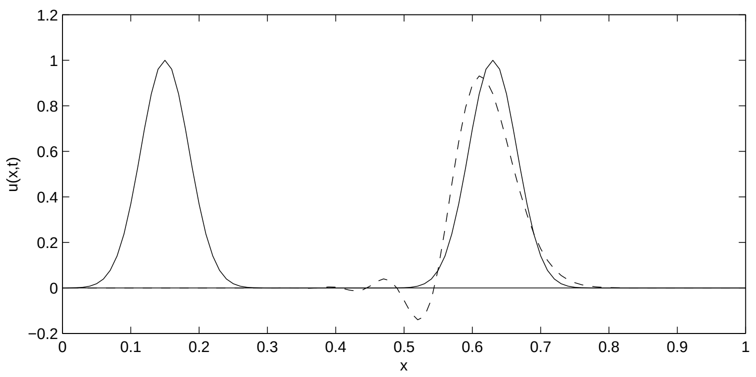
(Hint: $\sin x = x - x^3/3! + \ldots$, $\tan^{-1} x = x - x^3/3 + \ldots$)

$\Rightarrow$ The phase errors will grow when $|p| > 1$

$\Rightarrow$ The unconditional stability is paid with an increasing phase error

Crank-Nicolson scheme: Test example 1

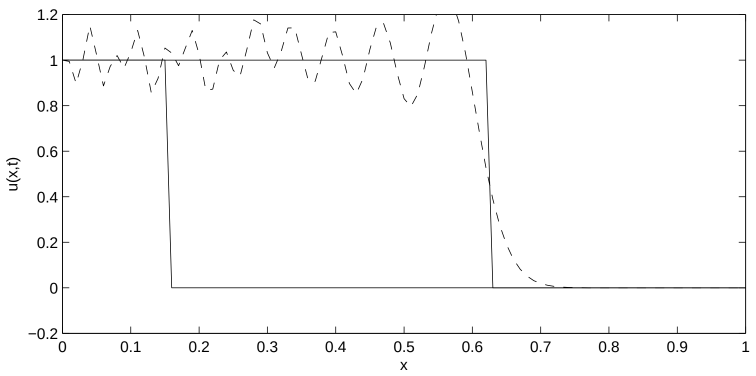
Initial condition: Gaussian pulse



Observation: Gaussian not well resolved plus some oscillations.

Crank-Nicolson scheme: Test example 2

Initial condition: Step function



Observation: Strong oscillations (due to missing artificial viscosity)

Summary of results: The different schemes for (AE)

Scheme	LTE	stable	$p = \pm 1$	Comments
Upwind	1st	$\|p\| \leq 1$	exact	Need FTBS for $a > 0$, FTFS for $a < 0$. Solutions smear out too much.
Leapfrog	2nd	$\|p\| \leq 1$	exact	Multi-level / Bad oscillations.
Lax-Wendroff	2nd	$\|p\| \leq 1$	exact	Best solution
CN	2nd	$\forall p$	Not exact	Implicit. Bad oscillations.

Second order equations: The wave equation

The wave equation

$$\begin{cases} u_{tt} = a^2 \Delta u \\ u(x,0) = h(x) & \text{1st IC}, \\ u_t(x,0) = H(x) & \text{2nd IC}. \end{cases}$$

is a prototype for 2nd order equations.

Physical meaning:

- Describes transmission of waves in different media, e.g. sound waves in air or water.
- The parameter a is the speed of the wave

Exact solution: D'Alembert solution

$$u(x,t) = F(x - at) + G(x + at)$$

Simplest numerical approximation: The CTCS scheme

Using a CTCS method leads to the 3-level scheme

$$\frac{w_j^{n+1} - 2w_j^n + w_j^{n-1}}{k^2} = \frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{h^2}$$

$$\Rightarrow \quad w_j^{n+1} = 2w_j^n - w_j^{n-1} + p^2(w_{j-1}^n - 2w_j^n + w_{j+1}^n), \quad p = ak/h \quad (*)$$

Fictitious point in time: For computing $n = 1$, we introduce an extra grid point at $t = -k$. Suppose we have at $t = 0$ ($n = 0$), i.e.,

$$u(x,0) = f(x) \quad \Rightarrow \quad w_j^0 = f_i$$

$$u_t(x,0) = g(x) \quad \Rightarrow \quad \frac{w_j^1 - w_j^{-1}}{2k} = g_i,$$

using central differences for the time derivative.

Now, write the CTCS scheme for $n = 0$ and eliminate the fictitious point $n = -1$ with previous equations, i.e.,

$$
\begin{aligned}
& w_j^1 = 2w_j^0 - w_j^{-1} + p^2(w_{j-1}^0 - 2w_j^0 + w_{j+1}^0) \\
\Rightarrow \quad & w_j^1 = 2f_j - w_j^{-1} + p^2(f_{j-1} - 2f_j + f_{j+1}) \\
\Rightarrow \quad & w_j^1 = 2f_j - w_j^1 + 2kg_i + p^2(f_{j-1} - 2f_j + f_{j+1}) \\
\Rightarrow \quad & w_j^1 = f_j + kg_i + \frac{1}{2}p^2(f_{j-1} - 2f_j + f_{j+1}), \quad j = 1, 2, \ldots, J.
\end{aligned}
$$

Exercise: Show that the LTE of the above CTCS scheme is

$$
\begin{aligned}
\text{LTE} &= \frac{1}{12}(k^2 u_{tttt} - a^2 h^2 u_{xxxx}) + O(k^4, h^4) \\
&= \frac{a^2}{12}(p^2 - 1)h^2 u_{xxxx} + O(h^4),
\end{aligned}
$$

i.e. it is 2nd order accurate.

Stability of the CTCS scheme $(*)$

Step 1: Insert the ansatz $w_j^n = \xi^n e^{i\omega j}$ into $(*)$, i.e.,

$$\begin{aligned}\xi^2 &= 2\xi - 1 + p^2\xi \underbrace{\left(e^{i\omega} - 2 + e^{-i\omega}\right)}_{=-4\sin^2(\omega/2)} \\ &= 2\xi[1 - 2p^2\sin^2(\omega/2)] - 1,\end{aligned}$$

hence $\xi^2 - 2[1 - 2p^2\sin^2(\omega/2)] + 1 = 0$.

Step 2: The roots of the quadratic equation are

$$\begin{aligned}\xi_\pm &= 1 - 2p^2\sin^2(\omega/2) \pm \sqrt{(1 - 2p^2\sin^2(\omega/2))^2 - 1} \\ &= 1 - 2p^2\sin^2(\omega/2) \pm \sqrt{4p^2\sin^2(\omega/2)[p^2\sin^2(\omega/2) - 1]}.\end{aligned}$$

Step 3: Discuss the discriminant $\sqrt{4p^2\sin^2(\omega/2)[p^2\sin^2(\omega/2)-1]}$:

- Case $p^2 < 1$: It holds that $4p^2\sin^2(\omega/2) \geq 0$ and $[p^2\sin^2(\omega/2)-1] \leq 0$ and therefore we get two complex roots

$$\xi_\pm = 1 - 2p^2\sin^2(\omega/2) \pm i2p\sin(\omega/2)\sqrt{1-p^2\sin^2(\omega/2)}$$

$$\Rightarrow \quad |\xi_\pm|^2 = [1-2p^2\sin^2(\omega/2)]^2 + 4p^2\sin^2(\omega/2)[1-p^2\sin^2(\omega/2)]$$
$$= 1 \quad \forall\omega.$$

- Case $p^2 = 1$: Gives $|\xi_\pm| = 1$

  $\Rightarrow$ Hence, the CTCS scheme $(*)$ is stable for $p^2 \leq 1$.

- Case $p^2 > 1$: Consider the best scenario, i.e., $\omega = \pi$,

$$\xi_\pm = 1 - 2p^2 \pm 2p\sqrt{p^2-1}$$

So $\quad \xi_- < 1 - 2p^2 < -1$ for $p^2 > 1$

$\Rightarrow \quad |\xi_-| > 1$ at $\omega = \pi$,

$\Rightarrow$ **Conclusion:** The CTCS scheme $(*)$ is von Neumann stable for

$$p^2 \leq 1.$$

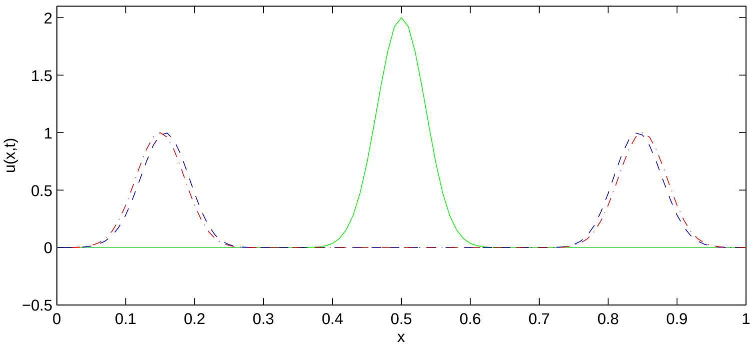
Exercise: Show that the phase error of the CTCS scheme $(*)$ is

$$\phi_{\pm} = \pm p\omega \left(1 - \frac{\omega^2}{24}(1 - p^2) + \dots \right).$$

This time we expect two solutions since the equation has waves travelling in both directions.

The CTCS method: Test example 1

Initial condition(s): Gaussian pulse in the center

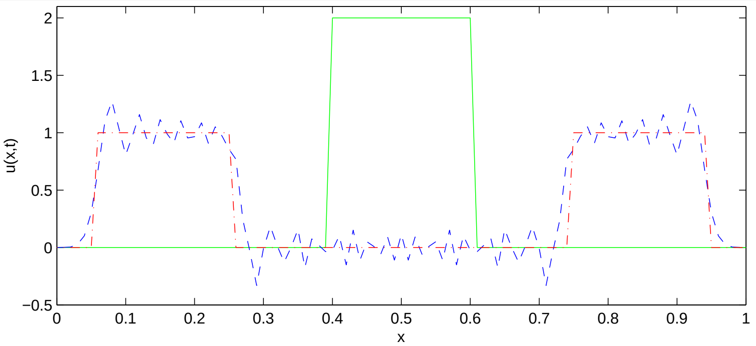


Observations: • Initial pulse splits into two smaller pulses travelling into opposite directions.

• Good agreement between dashed-dotted line (exact solution) and dashed line (numerical appr.).

The CTCS method: Test example 2

Initial condition(s): Square pulse in the center



Observations:
- Strong oscillations (as in the leapfrog scheme for **(AE)** [there is no damping (artificial viscosity)]