Numerical Methods for PDEs

Elliptic PDEs: Variational formulation/Linear FEM 1D

(Lecture 20, Week 7)

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Outline

Distributional and variational formulation

Consider the Poisson problem

$$\textbf{(PE)} \quad \begin{cases} -\mathrm{div}\,\left(\hat{\mathrm{I}}\nabla u\right) = f & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

Distributional formulation: *Multiply* **(PE)** *with a test function* $\varphi \in C_0^\infty(\Omega)$*, integrate over* Ω *and the integrate by parts to obtain*

$$0 = \int_\Omega \nabla u \nabla\varphi - f\varphi\, dx \qquad \forall \varphi \in C_0^\infty(\Omega)\,, \qquad \textbf{(WF)}$$

which is called distributional formulation of **(PE)**.

Variational Energy (VE) associated with (WF):

$$J(v) = \int_\Omega \frac{1}{2}(\nabla v)^2 - fv\, dx \qquad (\textbf{VE})$$

Variational Characterization (VC): The function $v \in C_0^\infty$, that minimizes $J(v)$, is the solution of **(PE)**.

Proof: Let $v = u + \delta u \in C_0^\infty$ be any function for which $J[v]$ is defined and which satisfies the boundary conditions of the ODE (i.e. $\delta u = 0$ at the boundaries). Then

$$
\begin{aligned}
\delta J &\equiv J[u+\delta u] - J[u] \\
&= \int_\Omega \left[\frac{1}{2}\left((\nabla u + \nabla \delta u)^2 - (\nabla u)^2\right) - f\left((u+\delta u) - u\right)\right] \mathrm{d}x \\
&= \int_\Omega \left[(\nabla u) . (\nabla \delta u) - f\delta u + \frac{1}{2}(\nabla \delta u)^2\right] dx
\end{aligned}
$$

We can simplify the first term in this expression by using integration by parts:

$$
\int_\Omega \delta u \Delta u dx = \int_{\partial\Omega} \delta u \nabla u \mathbf{n}\, do - \int_\Omega (\nabla \delta u) . (\nabla u) dx
$$

Hence we have

$$\delta J = \int_{\Omega} [-\Delta + f]\, \delta u \, dx + \int_{\partial\Omega} \delta u \nabla_{\mathbf{n}} u \, do + \mathcal{O}(\delta u^2)$$

Furthermore the second term in this expression is identically zero because $\delta u = 0$ on the boundary.

At a minimum δJ will vanish at leading order in δu. We see this can only happen if $d^2 u/dx^2 = f$ since δu is arbitrary inside Ω. In other words, if $u(x)$ is the function which minimises $J[u]$ then the function must satisfy $d^2 u/dx^2 = f$.

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Obtaining approximate solutions from **(VC)** in 1D

Making the ansatz of a Galerkin Approximation (GA) (truncated at $N > 0$)

$$v(x) \approx \sum_{k=1}^{N} c_k \phi_k(x), \qquad \textbf{(GA)}$$

where the $\phi_k(x)$ are a known set of *basis* functions and the c_k are unknown coefficients. Inserting this into **(VE)** gives

$$J[\mathbf{c}] = \int_\Omega \left[\frac{1}{2} \left(\sum_{k=1}^{N} c_k \frac{d\phi_k(x)}{dx} \right)^2 + \sum_{k=1}^{N} c_k f(x) \phi_k(x) \right] dx.$$

Now minimise over the c_k, that is, $\partial J/\partial c_j = 0,\ j = 1, \ldots, N$,

$$\int_\Omega \left[\frac{d\phi_j}{dx} \left(\sum_{k=1}^{N} c_k \frac{d\phi_k}{dx} \right) + f(x,y)\phi_j(x,y) \right] dx = 0$$

$$\Rightarrow \sum_{k=1}^{N} c_k \int_\Omega \left(\frac{d\phi_j}{dx}\frac{d\phi_k}{dx} \right) dx + \int_\Omega f(x)\phi_j(x)dx = 0$$

$$\Rightarrow \sum_{k=1}^{N} a_{jk} c_k + b_j = 0,$$

where

$$a_{j,k} = \int_\Omega \left(\frac{d\phi_j}{dx}\frac{d\phi_k}{dx} \right) dx, \quad b_j = \int_\Omega f(x)\phi_j(x).$$

In matrix form this is

$$A\mathbf{c} = -\mathbf{b}$$

where

$$A = \{a_{jk}\}, \quad \mathbf{b} = \{b_j\}.$$

By solving these equations for $\mathbf{c}$ we obtain the **(GA)**.

Piecewise linear basis functions: 1D case

Finite Element Method (FEM): The specific choice of the basis functions $\phi_j(x)$ determines the FEM.

Goal: $\phi_j(x)$ simple and supported on a small number of *elements*, which are line segments $[x_{j-1}, x_j], j = 1 \dots J$ in 1D.

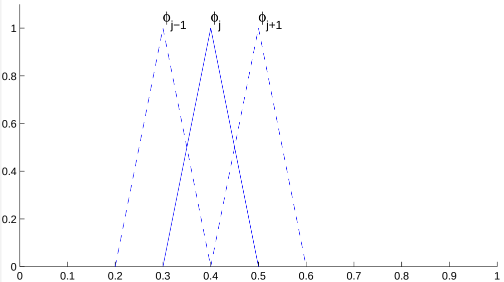


Figure: Piecewise linear *tent* functions, in the case $x_j = jh, h = 0.1$

Consider BCs such that $u(0) = u(1) = 0$, and assume $f(x) = f = \text{const}$.

Simplest basis functions: $\phi_j(x)$ *piecewise linear* (Figure)

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/(x_j - x_{j-1}), & x_{j-1} \le x \le x_j \\ (x_{j+1} - x)/(x_{j+1} - x_j), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise}. \end{cases}$$

and takes value 1 at $x = x_j$.

Since $u(x) = 0$ on the boundaries, in total there are $J - 1$ basis functions $\phi_j, j = 1, \ldots, J - 1$.

Calculate the right-hand side b_j

$$b_j = \int_0^1 f(x)\phi_j(x)\,dx = \int_0^1 f\phi_j(x)\,dx = f\int_{x_{j-1}}^{x_j} \frac{(x - x_{j-1})}{(x_j - x_{j-1})}\,dx$$

$$+ f\int_{x_j}^{x_{j+1}} \frac{(x - x_{j+1})}{(x_j - x_{j+1})}\,dx$$

$$= \frac{f}{2}\frac{(x - x_{j-1})^2}{(x_j - x_{j-1})}\Bigg|_{x=x_{j-1}}^{x=x_j} + \frac{f}{2}\frac{(x - x_{j+1})^2}{(x_j - x_{j+1})}\Bigg|_{x=x_j}^{x=x_{j+1}}$$

$$= \frac{f}{2}(x_j - x_{j-1}) + \frac{f}{2}(x_{j+1} - x_j).$$

For equally spaced nodes: $x_j - x_{j-1} = h$ for all j, and hence $b_j = fh$.

Calculate the matrix elements $a_{j,k}$

We note that ϕ_j' is a piecewise constant function

$$\phi_j'(x) = \begin{cases} 1/(x_j - x_{j-1}), & x_{j-1} \le x \le x_j \\ -1/(x_{j+1} - x_j), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise}. \end{cases}$$

First calculate $a_{jj} = \int \phi_j'(x)^2 \, dx$. The integrand is nonzero over both $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$.

$$a_{jj} = \int \phi_j'(x)^2 \, dx = \int_{x_{j-1}}^{x_j} \frac{1}{(x_j - x_{j-1})^2} \, dx + \int_{x_j}^{x_{j+1}} \frac{1}{(x_j - x_{j+1})^2} \, dx,$$
$$= \frac{1}{(x_j - x_{j-1})} + \frac{1}{(x_{j+1} - x_j)}$$

In the equally spaced case, we can simplify the final result to $a_{jj} = 2/h$.

Now consider $a_{j-1,j} = \int \phi'_{j-1}(x)\, \phi'_j(x)\, dx$. The integrand is nonzero only over $[x_{j-1}, x_j]$.

$$
\begin{aligned}
a_{j-1\,j} &= \int_{x_{j-1}}^{x_j} \phi'_{j-1}(x)\, \phi'_j(x)\, dx \\
&= \int_{x_{j-1}}^{x_j} \frac{-1}{(x_j - x_{j-1})} \cdot \frac{1}{(x_j - x_{j-1})}\, dx = -\frac{1}{(x_j - x_{j-1})}
\end{aligned}
$$

In the equally spaced case, this gives $a_{j-1\,j} = -1/h$.

A similar calculation shows that $a_{j,j+1} = -1/(x_{j+1} - x_j)$, which reduces to $-1/h$ in the equally spaced case.

So finally, in the equally spaced case, we have

$$\begin{pmatrix} 2/h & -1/h & 0 & \\ -1/h & 2/h & -1/h & 0 \\ \ddots & \ddots & \ddots & \ddots \\ & 0 & -1/h & 2/h \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{J-1} \end{pmatrix} = - \begin{pmatrix} fh \\ fh \\ \vdots \\ fh \end{pmatrix}.$$

Remark: After multiplying by $-h$ we get the central difference approximation **(CDA)**.

But for non-constant f: For piecewise linear

$$f(x) = \sum_{k=1}^{J-1} f_k \phi_k(x),$$

with $f(0) = f(1) = 0$ we find after some calculation that

$$b_j = \int_0^1 \phi_j(x) \left(\sum_{k=1}^{J-1} f_k \phi_k(x) \right) dx = \frac{h}{6} \left(f_{j-1} + 4 f_j + f_{j+1} \right),$$

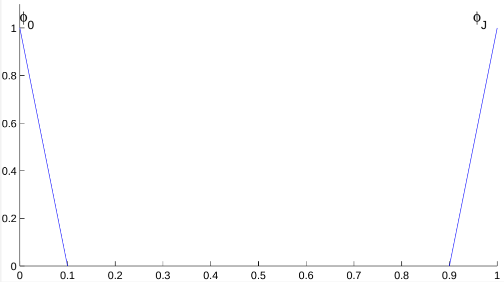
In the finite difference approach this would be just hf_j.

Nonzero boundary conditions $u(0) = \alpha$ & $u(1) = \beta$

Simply add extra “end” basis functions to the approximation of $u(x)$, i.e.,

$$v(x) \approx \tilde{v}(x) = \alpha\phi_0 + \sum_{k=1}^{J-1} c_k \phi_k(x) + \beta\phi_J$$

where ϕ_0 and ϕ_J are shown in the figure.



Neumann boundary condition

Consider the 1D Poisson equation

$$\textbf{(PE)} \quad \begin{cases} -\mathrm{div}\left(\hat{\mathrm{I}}\nabla u\right) = f \quad \text{in } \Omega \\ u(0) = 0\,, \\ u_x(1) = g\,, \end{cases}$$

Integration by part gives ($\phi(0) = 0$)

$$\int_\Omega \phi \left(\frac{d^2u}{dx^2}\right) dx = \left[\phi \frac{du}{dx}\right]_0^1 - \int_\Omega \left(\frac{d\phi}{dx}\right) \cdot \left(\frac{du}{dx}\right) dx\,,$$

and with the boundary conditions

$$\left[\phi \frac{du}{dx}\right]_0^1 = \underbrace{\phi(1)\frac{du(1)}{dx}}_{=\phi(1)g} - \underbrace{\phi(0)\frac{du(0)}{dx}}_{\phi(0)=0} = \phi(1)g\,.$$

The *variational formulation* of the Poisson problem reads now

$$\int_0^1 \frac{du(x)}{dx}\frac{d\phi(x)}{dx} dx = \phi(1)g - \int_0^1 f(x)\phi(x)dx.$$