Numerical Methods for PDEs

Elliptic PDEs: Linear FEM 2D

(Lecture 21, Week 7)

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Outline

1. Recall the distributional and variational formulations
2. Linear FEM 2D

Recall distributional and variational formulation

Consider the Poisson problem

$$\textbf{(PE)} \quad \begin{cases} -\mathrm{div}\left(\hat{\mathrm{I}}\nabla u\right) = f & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

Distributional formulation: *Multiply* **(PE)** *with a test function* $\varphi \in C_0^\infty(\Omega)$, *integrate over* Ω *and the integrate by parts to obtain*

$$0 = \int_\Omega \nabla u \nabla \varphi - f\varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega)\,, \qquad \textbf{(WF)}$$

which is called distributional formulation of **(PE)**.

Variational Energy (VE) associated with (WF):

$$J(v) = \int_\Omega \frac{1}{2}\left(\nabla v\right)^2 - fv \, dx \qquad (\textbf{VE})$$

Variational Principle:

Solving PDE **(PE)** $\overset{\textbf{Lecture 20}}{\Leftrightarrow}$ Minimising **(VE)**

Obtaining approximate solutions from **(VC)** in 2D

As in the 1D case, we make the ansatz of a Galerkin approximation

$$v(x,y) \approx \sum_{k=1}^{N} c_k \phi_k(x,y), \qquad \textbf{(GA)}$$

where the $\phi_k(x,y)$ are a known set of basis functions and the c_k are unknown coefficients. We then insert this into to get

$$J[\mathbf{c}] = \int\int_{\Omega} \Bigg[\frac{1}{2} \left(\sum_{k=1}^{N} c_k \frac{\partial \phi_k(x,y)}{\partial x} \right)^2 + \frac{1}{2} \left(\sum_{k=1}^{N} c_k \frac{\partial \phi_k(x,y)}{\partial y} \right)^2 + \sum_{k=1}^{N} c_k f(x,y) \phi_k(x,y) \Bigg] \, dx \, dy.$$

Now minimise over the c_k, for this we require that $\partial J/\partial c_j = 0,\ j = 1, \dots, N$

$$\int\int_\Omega \left[\frac{\partial \phi_j}{\partial x} \left(\sum_{k=1}^{N} c_k \frac{\partial \phi_k}{\partial x} \right) + \frac{\partial \phi_j}{\partial y} \left(\sum_{k=1}^{N} c_k \frac{\partial \phi_k}{\partial y} \right) + f(x,y)\phi_j(x,y) \right] dx\, dy = 0\,,$$

$$\sum_{k=1}^{N} c_k \int\int_\Omega \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_k}{\partial x} + \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_k}{\partial y} dxdy + \int\int_\Omega f(x,y)\phi_j(x,y)\, dx\, dy = 0\,,$$

$$\sum_{k=1}^{N} a_{j,k} c_k + b_j = 0\,,$$

where

$$a_{j,k} = \int\int_\Omega \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_k}{\partial x} + \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_k}{\partial y}\, dx\, dy, \quad b_j = \int\int_\Omega f(x,y)\phi_j(x,y)\, dx\, dy.$$

In matrix form this is given by

$$A\mathbf{c} = -\mathbf{b}$$

as in the 1D case, where

$$A = \left\{a_{j,k}\right\}, \qquad \mathbf{b} = \left\{b_j\right\}.$$

By solving these equations for $\mathbf{c}$ we obtain the **(GA).**

Triangular elements & linear basis functions

Finite Element Method (FEM) in 2D: We look for a FEM discretization to **(PE)**, i.e.,

$$\begin{cases} u_{xx} + u_{yy} = f & \text{in } \Omega := [0,1]^2 \,, \\ u = 0 & \text{on } \partial\Omega \,. \end{cases}$$

Ansatz: We use the Galerking approximation

$$u(x,y) \approx w(x,y) = \sum_{i=1}^{N} c_k \phi_k(x,y)$$

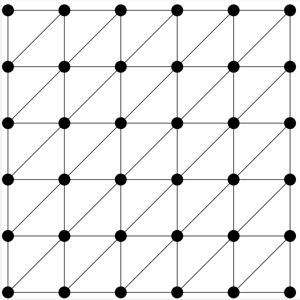
where the c_k satisfy the equation.

$$A\mathbf{c} = -\mathbf{b},$$

for $A = \{a_{j,k}\}$, and

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}.$$

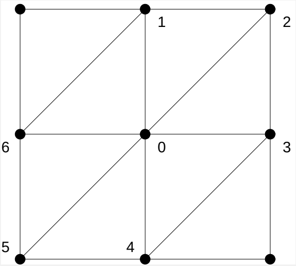
Triangulation: The simplest way to devide Ω into elements is to adopt a regular triangulation.



Nodes: Are the vertices of triangular elements

Labelling: We fix one node and label it '0' and its nearest neighbouring nodes we label by '1' to '6'.

We compute now: a_{jk} and b_j for $j = 0, \ldots, 6$ and $k = 0, \ldots, 6$



Choose $\phi_0(x, y)$ *linear* in x and y in each triangle $\Delta 012$, $\Delta 023$, i.e.

$$\phi_0(x, y) = a_0 + b_0 x + c_0 y\ ,$$

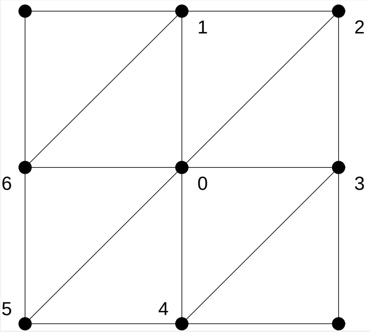
with a_0, b_0, c_0 *different* in each triangle. E.g. in $\Delta 012$, we choose a_0, b_0, c_0 s.t. $\phi_0(x_0, y_0) = 1, \phi_0(x_1, y_1) = 0, \phi_0(x_2, y_2) = 0$, where (x_k, y_k) are the coordinates of node k. Similarly in $\Delta 012$,

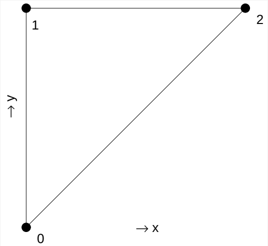
$$\phi_1(x, y) = a_1 + b_1 x + c_1 y,$$

with the constants a_1, b_1, c_1 chosen such that

$$\phi_1(x_0, y_0) = 0, \phi_1(x_1, y_1) = 1, \phi_1(x_2, y_2) = 0.$$

Compute a_{01}**:** Only the triangles $\Delta 012$ and $\Delta 016$ contribute.

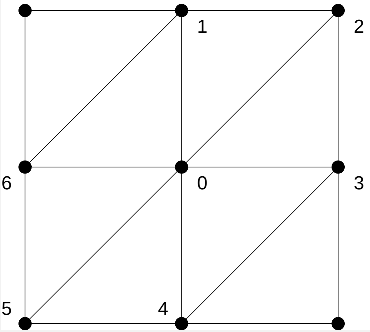


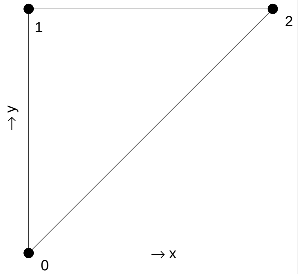


Consider $\Delta 012$**:** For equal sides h. Without loss of generality, we can move the origin to node 0.
We have

$$\phi_0 = \frac{1}{h}(h-y), \quad \phi_1 = \frac{1}{h}(y-x), \quad \phi_2 = \frac{1}{h}x,$$

$$\frac{\partial \phi_0}{\partial x} = 0, \quad \frac{\partial \phi_1}{\partial x} = -\frac{1}{h}, \quad \frac{\partial \phi_0}{\partial y} = -\frac{1}{h}, \quad \frac{\partial \phi_1}{\partial y} = \frac{1}{h}.$$

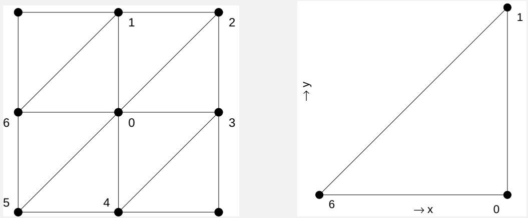




The contribution to a_{01} from $\Delta 012$ is

$$\iint_{\Delta 012} \left(0 \cdot \left(\frac{-1}{h} \right) + \left(\frac{-1}{h} \right) \cdot \left(\frac{1}{h} \right) \right) dx\, dy = -\frac{1}{h^2} \cdot \frac{h^2}{2} = -\frac{1}{2}$$

Consider $\Delta 016$:



A short calculation shows that

$$\phi_0 = \frac{1}{h}(h + x - y), \quad \phi_1 = \frac{1}{h}y, \quad \phi_6 = -\frac{1}{h}x,$$

$$\frac{\partial \phi_0}{\partial x} = \frac{1}{h}, \quad \frac{\partial \phi_1}{\partial x} = 0, \quad \frac{\partial \phi_0}{\partial y} = -\frac{1}{h}, \quad \frac{\partial \phi_1}{\partial y} = \frac{1}{h}.$$

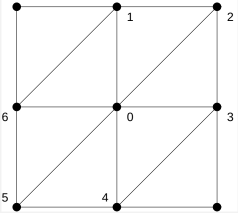
The contribution to a_{01} from $\Delta 016$ is

$$\iint_{\Delta 016} \left(\frac{-1}{h}\right) \cdot \left(\frac{1}{h}\right) dx\, dy = -\frac{1}{2}.$$

Hence $a_{01} = -\frac{1}{2} - \frac{1}{2} = -1$ and by symmetry,

$$a_{03} = a_{04} = a_{06} = a_{01} = -1.$$

Compute a_{02}**:** Only $\Delta 012$ and $\Delta 023$ contribute.

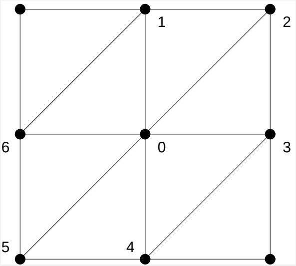


Consider $\Delta 012$**:** From the computation of a_{01} it follows that

$$\frac{\partial \phi_0}{\partial x} = 0, \quad \frac{\partial \phi_2}{\partial x} = \frac{1}{h}, \quad \frac{\partial \phi_0}{\partial y} = -\frac{1}{h}, \quad \frac{\partial \phi_2}{\partial y} = 0,$$

so the contribution to a_{02} is zero. By symmetry, the contribution from $\Delta 023$ is zero too. So $a_{02} = 0$, and by symmetry, $a_{05} = 0$ as well.

Compute a_{00}: $\Delta 012$, $\Delta 023$, $\Delta 034$, $\Delta 045$, $\Delta 056$, & $\Delta 016$ contribute.



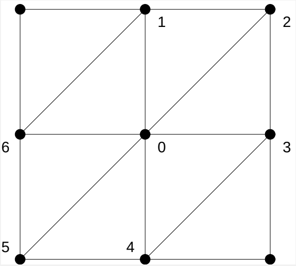
Consider $\Delta 012$**:** $\iint_{\Delta 012} \left(0^2 + \frac{1}{h^2}\right) dx\, dy = \frac{1}{2}$. and the same from $\Delta 023$, $\Delta 045$, and $\Delta 056$ by symmetry.

Consider $\Delta 016$**:** $\iint_{\Delta 016} \left(\frac{1}{h^2} + \frac{1}{h^2}\right) dx\, dy = 1$, and the same from $\Delta 034$ by symmetry.

The total is

$$a_{00} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 + 1 = 4.$$

Compute b_0: $\Delta 012$, $\Delta 023$, $\Delta 034$, $\Delta 045$, $\Delta 056$, & $\Delta 016$ contribute.

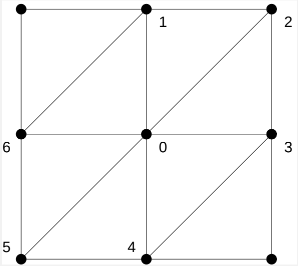


Consider $\Delta 012$**:**

$$f \int_0^h \int_{x=0}^{x=y} \frac{1}{h}(h-y)\,dx\,dy = \frac{f}{h}\int_0^h y(h-y)dy$$

$$= \frac{f}{h}\int_0^h hydy - \frac{f}{h}\int_0^h y^2 dy = \left[\frac{fy^2}{2} - \frac{fy^3}{3h}\right]_0^h = \frac{fh^2}{6}.$$

By symmetry, there is the same contribution from $\Delta 023$, $\Delta 045$, $\Delta 056$.



Consider $\Delta 016$**:**

$$\frac{f}{h}\int_0^h \int_{x=y-h}^{x=0} (h+x-y)\,dx\,dy = \frac{f}{2h}\int_0^h (h+x-y)^2\Big|_{x=y-h}^{x=0} dy$$

$$= \frac{f}{2h}\int_0^h (h-y)^2 dy = -\frac{f}{6h}(h-y)^3\Big|_0^h = \frac{fh^2}{6},$$

and the same from $\Delta 034$ by symmetry. So the total contribution to b_0 is

$$b_0 = 6 \times \frac{fh^2}{6} = h^2 f.$$

So finally the equation for c_0 from node 0 is

$$-(c_1 + c_3 + c_4 + c_6) + 4c_0 = -h^2 f$$

or

$$(c_1 + c_3 + c_4 + c_6) - 4c_0 = h^2 f\,.$$

This is the same, in this simple case, as in the Central Difference approximation of

$$u_{xx} + u_{yy} = f.$$

Remark: Note that FEM allows us to cover a more complicated area with triangles and hence we are able to deal with odd shapes.