Numerical Methods for PDEs

Elliptic PDEs: FEM 2D/Sobolev spaces/Stability/Error estimates

(Lecture 23, Week 8)

Markus Schmuck

Department of Mathematics and Maxwell Institute for Mathematical Sciences
Heriot-Watt University, Edinburgh

Edinburgh, March 5, 2014

Outline

1. Rectangular elements and bilinear basis functions

2. Stability and error estimates for FEM

Rectangular elements and bilinear basis functions

Rectangular domains: Can be covered by rectangular elements with *bilinear basis functions*

$$\phi_0(x,y) = \frac{1}{h^2}(h-x)(h-y),$$
$$\phi_1(x,y) = \frac{1}{h^2}(h-x)y$$
$$\phi_2(x,y) = \frac{1}{h^2}xy,$$
$$\phi_3(x,y) = \frac{1}{h^2}x(h-y)$$

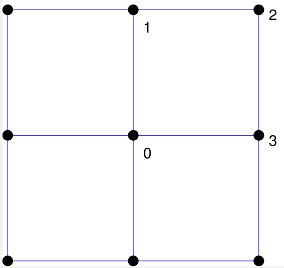

Figure : In the square 0123, we define 4 basis functions ϕ_k each taking value 1 at node k and 0 at the other nodes

We can
now repreat the previous calculations with these new basis functions.

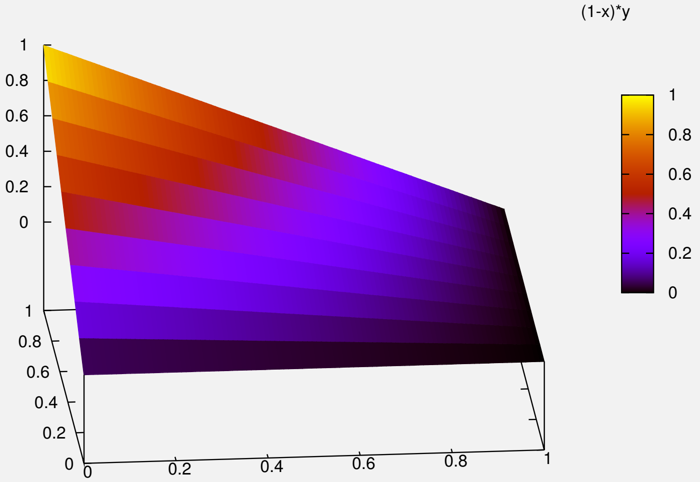

Figure : Plot of the ϕ_1 bilinear basis function for $h = 1$.

Stability and error estimates for FEM

Goal: Derivation of error estimates between the exact solution u for elliptic problems

$$(\nabla u, \nabla \phi) + (u, \phi) = (f, \phi) \quad \forall \phi \in V,$$

and the finite element solution w

$$(\nabla w, \nabla \phi) + (w, \phi) = (f, \phi) \qquad \forall \phi \in V_h.$$

Assumptions: f is constant, Ω is the unit squar, and $u = 0$ on the boundary
Notation: (ϕ, ψ) means

$$(\phi, \psi) = \int_\Omega \phi \psi \, d\mathbf{x} \, .$$

V_h is the space of piecewise linear finite element basis functions.

Sobolev spaces

The $L_2(\Omega)$ Hilbert space is a space of integrable real-valued functions $\phi : \Omega \to \mathbb{R}$ for which the following norm (or integral) is bounded:

$$\|\phi\|^2_{L_2(\Omega)} = \int_\Omega \phi^2 \mathrm{d}\Omega < \infty.$$

We call the above norm the L_2-norm and for simplicity use the notation without the subscript, i.e., $\|\cdot\| \equiv \|\cdot\|_{L_2(\Omega)}$.
The functions ϕ belonging to the space $L_2(\Omega)$ are also called $L_2(\Omega)$-integrable.
The space of L_2-integrable functions $\phi \in L_2(\Omega)$ is equipped with the following scalar product

$$(\psi, \phi) = \int_\Omega \psi\phi \, \boldsymbol{dx} < \infty.$$

Note that the scalar product satisfies

$$(\phi, \phi) = \|\phi\|^2.$$

Further the L_2-scalar product satisfies the so-called Cauchy-Schwartz inequality

$$|(\phi, \psi)| \leq \|\phi\| \, \|\psi\| \, .$$

The space $H^1(\Omega)$ (also called first Sobolev space) is a space of functions which are bounded in the following norm

$$\|\phi\|^2_{H^1(\Omega)} = \|\phi\|^2 + \|\nabla\phi\|^2 < \infty.$$

Thus the $H^1(\Omega)$ space contains L_2-integrable functions with L_2-integrable first order derivatives.

The space $H^2(\Omega)$ (also called second Sobolev space) is the space of functions which are bounded in the following norm

$$\|\phi\|^2_{H^2(\Omega)} = \|\phi\|^2 + \|\nabla\phi\|^2 + \|\nabla^2\phi\|^2 < \infty. \qquad (*)$$

Thus the $H^2(\Omega)$ space contains L_2-integrable functions with L_2-integrable first and second order derivatives. Note, that the norm **(∗)** is a slightly simplified version of the true H^2-norm, but the two norms are equivalent for simple domains Ω, such as considered here. Further we define the H^1 and H^2 semi-norms as

$$|\phi|_{H^1} = \|\nabla\phi\| \qquad \text{and} \qquad |\phi|_{H^2} = \|\nabla^2\phi\|,$$

respectively.

The H^1 and H^2 spaces are more general analogues of the spaces $C^1(\Omega)$ (smooth functions with continuous first order derivatives) and $C^2(\Omega)$ (smooth functions with continuous second order derivatives). We also have that $C^1(\Omega) \subset H^1$, $C^2(\Omega) \subset H^2$ and $H^2 \subset H^1 \subset L_2(\Omega)$. The following inequality (which is a combination of the Cauchy-Schwarz and Young's ($|a|\,|b| \le C_\epsilon |a|^2 + \epsilon |b|^2$ for $a, b \in \mathbb{R}^d$, $d \ge 1$) inequalities) will be useful

$$|(\phi, \psi)| \le C_\epsilon \|\phi\|^2 + \epsilon \|\psi\|^2 \qquad \forall \phi, \psi \in L^2,$$

where $\epsilon > 0$ is a arbitrary small positive constant, and C_ϵ is a positive constant depending on ϵ (C_ϵ grows for $\epsilon \to 0$). Note that for $\epsilon = 1/2$ the above inequality becomes

$$(\phi, \psi) \le \frac{1}{2}\|\phi\|^2 + \frac{1}{2}\|\psi\|^2 \qquad \forall \phi, \psi \in L^2.$$

Interpolation

We define the interpolation operator $I^h : C(\Omega) \to V^h$ from the space of continuous function to the space V^h of piecewise linear functions such that

$$I^h\phi(\mathbf{x}_k) = \phi(\mathbf{x}_k),$$

for all points $\mathbf{x}_k$ that belong to the finite element mesh. Thus, for a given continuous function ϕ, the interpolation operator produces a piecewise linear function $I^h\phi$ that is equal to the original function at all mesh points. For a function ϕ, the function $I^h\phi$ is called the interpolant of ϕ.

The error between a function $\phi \in H^2$ and its piecewise linear interpolant $I^h\phi \in V^h$ can be estimated from the following "interpolation estimate"

$$\|\phi - I^h\phi\|_{H^1} \leq Ch|\phi|_{H^2} \qquad \forall\phi \in H^2,$$

where h is the mesh size and C is a fixed positive constant independent of h. We can see that $I^h\phi \to \phi$ as $h \to 0$, i.e. the error gets smaller for finer meshes.

Stability

The finite element solution w satisfies

$$(\nabla w, \nabla \phi) + (w, \phi) = (f, \phi) \qquad \forall \phi \in V_h. \tag{1}$$

The above equality is valid for any $\phi \in V^h$, thus we can take $\phi = w \in V^h$. Then (1) becomes

$$(\nabla w, \nabla w) + (w, w) = (f, w)\,,$$

which is equivalent to

$$\|w\|_{H^1}^2 = (f, w).$$

The RHS can be estimated using Hölder & Young's inequality with $\epsilon = 1/2$ as

$$(f, w) \leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\|w\|^2.$$

After combining the above calculations we arrive at

$$\|w\|_{H^1}^2 = \|w\|^2 + \|\nabla w\|^2 = \frac{1}{2}\|f\|^2 + \frac{1}{2}\|w\|^2.$$

Next we subtract $\frac{1}{2}\|w\|^2$ from the above equation and get

$$\|w\|^2 + \frac{1}{2}\|\nabla w\|^2 \leq \|f\|^2,$$

which is equivalent to

$$\|w\|_{H^1}^2 \leq C\|f\|^2,$$

for some (fixed) positive constant C independent of f, w, h.
Thus, if $f \in L^2$, we have just shown that finite element solution w is bounded in H^1-norm by a constant that depends on f and Ω (but not on h). Thus, the finite element solution is stable in the $V^h \approx H^1$ space for any h.

Error estimates

The exact solution u satisfies

$$(\nabla u, \nabla \phi) + (u, \phi) = (f, \phi) \qquad \forall \phi \in H^1(\Omega). \tag{2}$$

The finite element solution satisfies

$$(\nabla w, \nabla \phi) + (w, \phi) = (f, \phi) \qquad \forall \phi \in V^h \subset H^1(\Omega). \tag{3}$$

We subtract (3) from (2) and for all $\phi \in V^h$ we have

$$(\nabla e_h, \nabla \phi) + (e_h, \phi) = 0,$$

where $e_h = u - w$. Next, we set $\phi = I^h u - w$ and get

$$\begin{aligned}
&(\nabla e_h, \nabla (I^h u - w)) + (e_h, I^h u - w) \\
&= (\nabla e_h, \nabla (I^h u \mp u - w)) + (e_h, I^h u \mp u - w) \\
&= \|\nabla e_h\|^2 + \|e_h\|^2 + (\nabla e_h, \nabla (I^h u - u)) + (e_h, (I^h u - u)) \\
&= 0
\end{aligned}$$

After putting the last two term to the RHS we obtain

$$\|\nabla e_h\|^2 + \|e_h\|^2 = (\nabla e_h, \nabla(u - I^h u)) + (e_h, (u - I^h u)).$$

Next, we apply the inequality Hölder's & Young's ineq. with $\epsilon = 1/2$

$$\|\nabla e_h\|^2 + \|e_h\|^2 \leq \frac{1}{2}\left(\|\nabla e_h\|^2 + \|e_h\|^2\right) + \frac{1}{2}\|u - I^h u\|_{H^1}^2.$$

We move the first two terms on the RHS to the LHS and use the interpolation inequality to get

$$\frac{1}{2}(\|\nabla e_h\|^2 + \|e_h\|^2) \leq \|u - I^h u\|_{H^1}^2 \leq Ch^2 |u|_{H^2(\Omega)}^2.$$

Which, after taking a square root, proves

$$\|e_h\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}.$$

We have just shown that if the exact solution $u \in H^2$ (i.e., $|u|_{H^2(\Omega)}$ is bounded) then

$$\|u - w\|_{H^1} \approx O(h).$$