Numerical Methods for PDEs

A Finite Difference Scheme for the Heat Equation

(Lecture 3, Week 1)

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Outline

1. The FTCS scheme
2. Example with exact solution
3. Example of an unstable scheme

Simple Finite Difference scheme

Goal: Find a suitable approximation w_j^n of the exact solution $u(x_j, t_n) = u_j^n$ at the mesh points (nodes) (x_j, t_n) of the dimensionless equation

$$\begin{cases} u_t - u_{xx} = 0 & x \in (0,1)\,, \\ u(x,0) = F(x)\,, & t = 0\,, \\ u(0,t) = \alpha(t)\,, & t > 0\,, \\ u(1,t) = \beta(t)\,, & t > 0\,. \end{cases}$$

The simplest discrete approximation is the solution w_j^n of

$$D_t^+ w_j^n = D_x^2 w_j^n\,,$$

at all interior points (x_j, t_n), $0 < j < J$, $n > 0$.

Remark. *Note that*

$$u_t(x_j, t_n) \approx D_t^+ u_j^n\,, \qquad u_{xx}(x_j, t_n) \approx D_X^2 u_j^n\,,$$

and hence $D_t^+ u_j^n \approx D_x^2 u_j^n$.

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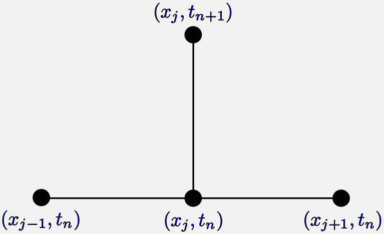

Figure: Explicit finite difference scheme.

and after defining $r := \frac{k}{h^2}$ and re-arranging, we get the

$$\text{FTCS scheme:} \qquad w_j^{n+1} = rw_{j-1}^n + (1-2r)w_j^n + rw_{j+1}^n .$$

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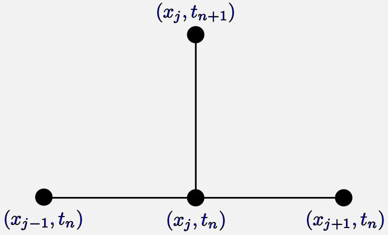

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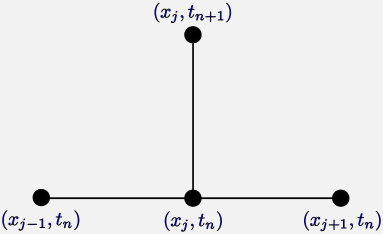

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The FTCS scheme

Algorithm:

1. Choose J and r; then calculate $h = 1/J$ and $k = rh^2$.

2. Calculate (initial condition) $w_j^0 = F(x_j)$ and set $n = 0$.

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for $j = 1, \ldots, J-1$ and set (the boundary conditions)

$$w_0^{n+1} = \alpha(t_{n+1}) \quad \text{and} \quad w_J^{n+1} = \beta(t_{n+1})\,.$$

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by the FTCS scheme with $r = 0.4$, $J = 2$ and two time steps.

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1. $J = 2$, $h = 1/J = 1/2$, and $k = rh^2 = 0.4 \cdot 1/4 = 0.1$.
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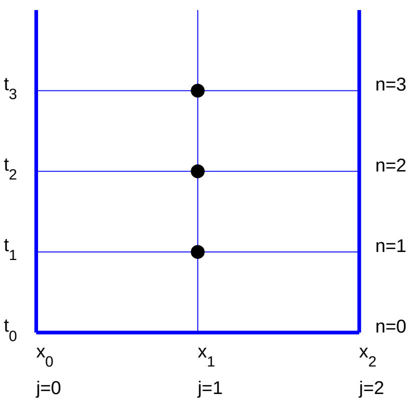

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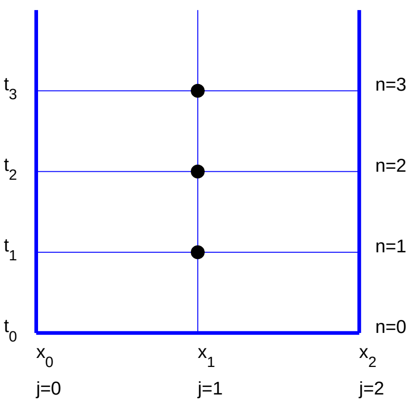

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leads to problems (due to the discontinuous derivative), see Figure below.

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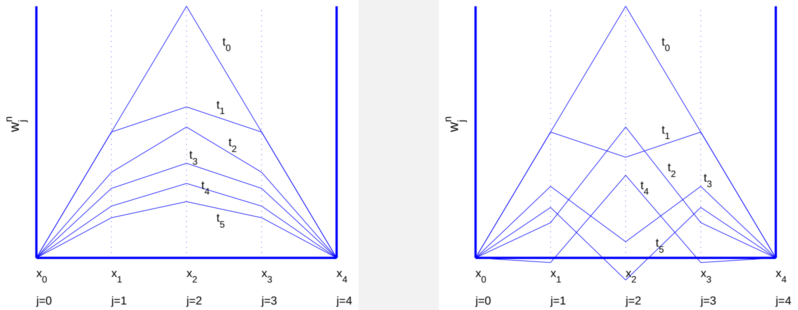

Figure: **Left:** $r = 0.4$, w_j^n slowly decreases to zero (as expected). **Right:** $r = 0.6$, w_j^n shows spatial oscillations.

Problems in Figure on the Right:

1. Already for small (times) t_n, the temperature w_j^n becomes negative.
2. Spatial oscillations become unbounded

$$w_j^n \to \infty \quad \text{for} \quad n \to \infty$$

in contrast to the exact solution

$$u \to 0 \quad \text{for} \quad t \to \infty\,.$$

A numerical scheme with this sort of bad behavior (e.g. nonphysical/unbounded oscillations) is called *unstable*.

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