Numerical Methods for PDEs

Stability of Finite Difference Schemes

(Lecture 5, Week 2)

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Outline

1. Stability determined by eigenvalues

2. The von Neumann/Fourier method

Stability of the FTCS scheme: The eigenvalue method

Recall: Matrix form of the FTCS scheme

$$\mathbf{w}^{n+1} = S^{n+1}\mathbf{w}^0 .$$

Some facts about eigenvalues:

1. If λ is an eigenvalue of S and $\mathbf{e}$ a corresponding eigenvector, i.e.,

$$S\mathbf{e} = \lambda\mathbf{e} ,$$

then for $n \to \infty$

$$|S^n\mathbf{e}| = |\lambda^n\mathbf{e}| \to \infty , \quad \text{if } |\lambda| > 1 ,$$

where $|\cdot|$ is the Euclidean norm $|x| := (|x_1|^2 + \cdots + |x_{J-1}|^2)^{1/2}$.

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where $|\cdot|$ is the Euclidean norm $|x| := (|x_1|^2 + \cdots + |x_{J-1}|^2)^{1/2}$.

2. If all eigenvalues of S satisfy $|\lambda| \leq 1$, then for $n \to \infty$ it holds that

$$\begin{cases} |S^n \mathbf{z}| \to 0\,, & \text{if} \quad (|\lambda| < 1) \\ |S^n \mathbf{z}| < \infty\,, & \text{if} \quad (|\lambda| = 1) \end{cases}$$

for all vectors z.

3. A tri-diagonal matrix

$$S := \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & c & \ddots & \ddots & \\ & & \ddots & a & b \\ & & & c & a \end{bmatrix} \in \mathbb{R}^{(J-1)\times(J-1)}$$

has the eigenvalues

$$\lambda_s = a + 2\sqrt{bc}\cos\left(\frac{\pi s}{J}\right)$$

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Hence, using 2. and 3. from the previous slide gives

$$-1 \leq a + 2\sqrt{bc}\cos\left(\frac{s\pi}{J}\right) \leq 1\,,$$

and with $\cos(2x) = 1 - 2\sin^2(x)$, $a = 1 - 2r$, and $a = c = r := k/h^2$ the right-hand inequality is always true, since

$$-2\sin^2(s\pi/(2J)) \leq 0\,.$$

The left-hand inequality leads to

$$1 - 4r\sin^2\left(\frac{s\pi}{2J}\right) \geq -1$$

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Remarks.

(i) This way of analysing the stability of a scheme is not easily generalized since it involves finding the eigenvalues of the corresponding S-matrix.

(ii) The condition $|\lambda_s| \leq 1$ only guarantees stability because S is symmetric (true in general for parabolic equations but not for hyperbolics).

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von Neumann stability

Recall: The FTCS scheme for the heat equation

$$D_t^+ w_j^n = D_x^2 w_j^n \,, \qquad j = 1, \ldots, J-1 \,,$$

that is, $w_j^{n+1} = (1-2r)w_j^n + r(w_{j+1}^n + w_{j-1}^n)$.

Basic idea: Consider a harmonic initial perturbation

$$w_j^0 = e^{i\lambda x_j} = e^{i\lambda jh}, \qquad \omega \in \mathbb{R} \,,$$

which evolves in time as

$$w_j^n = \xi^n e^{i\lambda jh} \,,$$

while we neglect boundary conditions. Then, stability requires that

$$|\xi| \leq 1 \,,$$

where ξ is called *amplification factor*. Sometimes, we set $\omega := \lambda h$.

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von Neumann stability of the FTCS scheme

Insert $w_j^n = \xi^n e^{i\lambda jh}$ into the FTCS scheme

$$\begin{aligned}
\xi^{n+1} e^{i\lambda jh} &= \xi^n e^{i\lambda jh} \left(1 - 2r + r\left(e^{i\lambda h} + e^{-i\lambda h}\right)\right) \\
&= \xi^n e^{i\lambda jh} \left(1 + 2r\left(\cos(\lambda h) - 1\right)\right) \\
&= \xi^n e^{i\lambda jh} \left(1 + 2r\left(-2\sin^2(\lambda h/2)\right)\right)
\end{aligned}$$

where we used $\cos(2x) = 1 - 2\sin^2(x)$. After dividing both sides by $\xi^n e^{i\lambda jh}$, we get

$$\xi = 1 - 4r\sin^2(\lambda h/2)\,,$$

for which $w_j^n = \xi^n e^{i\lambda jh}$ is a solution to the FTCS scheme.

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von Neumann stability and unstable solutions

Definition. *The scheme is said to be unstable, if $|\xi| > 1$ since then $\left|w_j^n\right| = |\xi^n| \to \infty$ as $n \to \infty$.*

The scheme is said to be von Neumann stable, if $|\xi| \leq 1$.

Claim: The FTCS scheme is von Neumann stable, if $r := k/h^2 \leq 1/2$.

Proof: The requirement $|\xi| \leq 1$ reads

$$-1 \leq 1 - 4r\sin^2(\lambda h/2) \leq 1\,,$$

where the RHS is obviously satisfied and the LHS gives

$$r\sin^2(\lambda h/2) \leq 1/2\,,$$

which proves in the worst case $\lambda h = \pi$ the claim. □

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Summary - von Neumann stability analysis

1. **Methodology:** Substitute $w_j^n = \xi^n \exp(i\omega j)$ into the difference scheme, and solve for ξ in terms of $\omega := h\lambda, r$, etc.
2. Determine if the amplification factor ξ has modulus ≤ 1 for all values of $|\omega| \leq \pi$. If this is so for all values of r we have *unconditional stability*.
3. If $|\xi| \leq 1$ for some range of r, we say the scheme is *von Neumann stable* for r in the stated range, otherwise the scheme is *unstable*.

Notes:

- **von Neumann stability:** i) Necessary but not sufficient (e.g. difference schemes with 3 or more time levels). ii) Difficult for nonzero boundary conditions. iii) Gives often useful results even if its application is not fully justified.
- **Exponentially in time increasing exact solutions:** Requires the *modified* von Neumann stability condition

$$|\xi| \leq 1 + Kk$$

for some positive K in the limit of small k.

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Summary - von Neumann stability analysis

1. **Methodology:** Substitute $w_j^n = \xi^n \exp(i\omega j)$ into the difference scheme, and solve for ξ in terms of $\omega := h\lambda, r$, etc.
2. Determine if the amplification factor ξ has modulus ≤ 1 for all values of $|\omega| \leq \pi$. If this is so for all values of r we have *unconditional stability*.
3. If $|\xi| \leq 1$ for some range of r, we say the scheme is *von Neumann stable* for r in the stated range, otherwise the scheme is *unstable*.

Notes:

- **von Neumann stability:** i) Necessary but not sufficient (e.g. difference schemes with 3 or more time levels). ii) Difficult for nonzero boundary conditions. iii) Gives often useful results even if its application is not fully justified.
- **Exponentially in time increasing exact solutions:** Requires the *modified* von Neumann stability condition

$$|\xi| \leq 1 + Kk$$

for some positive K in the limit of small k.

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1. Is the eigenvalue method generally applicable for stability of numerical schemes?

2. When is the FTCS scheme unstable and can you derive this criterion by the von Neumann method?

3. Is the von Neumann stability criterion a sufficient condition for stability?

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