Numerical Methods for PDEs

Stability of the θ-Method and Extensions

(Lecture 7, Week 3)

Markus Schmuck

Department of Mathematics and Maxwell Institute for Mathematical Sciences
Heriot-Watt University, Edinburgh

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Outline

1. Stability of the θ-method
2. Matrix form of the θ-method

von Neumann analysis

For $r = k/h^2$, the θ-method reads

$$-\theta r w_{m-1}^{n+1} + (1+2\theta r) w_m^{n+1} - \theta r w_{m+1}^{n+1} = (1-\theta) r w_{m-1}^{n} + (1-2(1-\theta)r) w_m^{n} + (1-\theta) r w_{m+1}^{n}\,.$$

Then, substitute $w_m^n = \xi^n e^{im\omega}$ and simplify

$$-\theta r e^{i(m-1)\omega}\xi^{n+1} + (1+2\theta r) e^{im\omega}\xi^{n+1} - \theta r e^{i(m+1)\omega}\xi^{n+1} = (1-\theta) r e^{i(m-1)\omega}\xi^{n} + (1-2(1-\theta)r) e^{im\omega}\xi^{n} + (1-\theta) r e^{i(m+1)\omega}\xi^{n}$$

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$$\Rightarrow -\theta r e^{-i\omega} \xi + (1+2\theta r) \xi - \theta r e^{i\omega} \xi \\ = (1-\theta) r e^{-i\omega} + (1 - 2(1-\theta) r) + (1-\theta) r e^{i\omega}$$

(taking out factors $e^{im\omega}$ and ξ^n)

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$$\Rightarrow \xi - \xi \theta r (e^{i\omega} - 2 + e^{-i\omega}) = 1 + (1-\theta) r (e^{i\omega} - 2 + e^{-i\omega})$$

Using

$$e^{i\omega} - 2 + e^{-i\omega} = -2(1 - \cos(\omega)) = -4\sin^2(\omega/2)$$

in the above equation gives

$$\xi + 4\xi\theta\sin^2(\omega/2)r = 1 - 4(1-\theta)r\sin^2(\omega/2)$$

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$$\Rightarrow \xi = \frac{1 - 4(1-\theta)r\sin^2(\omega/2)}{1 + 4\theta r\sin^2(\omega/2)}$$

We need $|\xi| \leq 1$ for stability for all $\omega \in [-\pi, \pi]$. Since ξ is clearly real in this case this means we require $-1 \leq \xi \leq 1$. Now

$$\xi = \frac{1 + 4\theta r\sin^2(\omega/2) - 4r\sin^2(\omega/2)}{1 + 4\theta r\sin^2(\omega/2)} = 1 - \frac{4r\sin^2(\omega/2)}{1 + 4\theta r\sin^2(\omega/2)}$$

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so $\xi \le 1$ is satisfied.

It leaves to consider the case $\xi \geq -1$. This is (on multiplying through by the denominator)

$$-1 - 4\theta r \sin^2(\omega/2) \leq 1 - 4(1-\theta) r \sin^2(\omega/2)$$
$$\Rightarrow 2(1-2\theta) r \sin^2(\omega/2) \leq 1$$

(i) If $\theta \geq 1/2$, this last inequality will clearly hold for *all* r.

(ii) If $\theta < 1/2$, we need in the worst case ($\omega = \pi$) that

$$r \leq \frac{1}{2(1-2\theta)}.$$

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Stability result for the θ-scheme

Summary: The θ-scheme is stable

(i) for *all* r, if $\theta \geq 1/2$,

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Matrix form of the θ-method

We can write the θ-method in much the same form as the BTCS scheme,

$$S\mathbf{w}^{n+1} = M\mathbf{w}^n + (1-\theta)\mathbf{b}^n + \theta\mathbf{b}^{n+1}$$

where

$$S = \begin{pmatrix} 1+2\theta r & -\theta r & 0 & \dots & \\ -\theta r & 1+2\theta r & -\theta r & 0 & \dots \\ 0 & -\theta r & 1+2\theta r & -\theta r & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & 0 & -\theta r & 1+2\theta r \end{pmatrix},$$

$$\mathbf{w}^n = \begin{pmatrix} w_1^n \\ w_2^n \\ \vdots \\ \vdots \\ w_{J-1}^n \end{pmatrix}, \quad \mathbf{b}^n = \begin{pmatrix} r\alpha(t_n) \\ 0 \\ \vdots \\ 0 \\ r\beta(t_n) \end{pmatrix},$$

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$$M = \begin{pmatrix} 1-2(1-\theta)r & (1-\theta)r & 0 & \dots & \\ (1-\theta)r & 1-2(1-\theta)r & (1-\theta)r & 0 & \dots \\ 0 & (1-\theta)r & 1-2(1-\theta)r & (1-\theta)r & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & 0 & (1-\theta)r & 1-2(1-\theta)r \end{pmatrix},$$

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(i) Set $\mathbf{q} = M\mathbf{w}^n + (1-\theta)\mathbf{b}^n + \theta\mathbf{b}^{n+1}$.

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The matrix S is tridiagonal (if $\theta > 0$), so solving (ii) is fairly quick and easy. This is still more work than solving the explicit FTCS scheme ($\theta = 0$) but not much more.

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$0 < \theta < 1/2$	$O(k, h^2)$	implicit	$r \leq 1/2(1 - 2\theta)$
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(Note the accuracy of some schemes can be increased by choosing a special value for r).

Remark. 1. FTCS scheme easy to apply (because it is explicit), but the time step constraint for stability requires $k \leq 1/2h^2$, such that for h small, we have k very small.

2. The θ-method for $\theta > 1/2$ allows a larger time step for stability (but not too large, otherwise the LTE gets big), and hence can require less overall computing.

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1. Can you apply the von Neumann stability to the θ-method?
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