Numerical Methods for PDEs

Extensions and Generalisations

(Lecture 8, Week 3)

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Outline

1. Application of these numerical schemes to more general PDEs

2. More general boundary conditions

Extension to more general PDEs

Consider the PDE $u_t = u_{xx} + c_1 u_x + c_2 u$.

The FTCS scheme reads

$$\frac{F_t}{k} w_j^n = \frac{\delta_x^2}{h^2} w_j^n + c_1 \frac{D_x}{2h} w_j^n + c_2 w_j^n$$

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\Rightarrow \quad & w_j^{n+1} = (r - c_1 p) w_{j-1}^n + (1 - 2r + c_2 k) w_j^n + (r + c_1 p) w_{j+1}^n ,
\end{aligned}
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The θ-method becomes

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\frac{F_t}{k} w_j^n = (1-\theta) &\left(\frac{\delta_x^2}{h^2} w_j^n + c_1 \frac{D_x}{2h} w_j^n + c_2 w_j^n \right) \\
&+ \theta \left(\frac{\delta_x^2}{h^2} w_j^{n+1} + c_1 \frac{D_x}{2h} w_j^{n+1} + c_2 w_j^{n+1} \right)
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$$\Rightarrow -\theta(r - c_1 p) w_{j-1}^{n+1} + (1 + 2\theta r - \theta k c_2) w_j^{n+1} - \theta(r + c_1 p) w_{j+1}^{n+1} = (1-\theta)(r - c_1 p) w_{j-1}^n + (1 + (kc_2 - 2r)(1-\theta)) w_j^n + (1-\theta)(r + c_1 p) w_{j+1}^n.$$

Conclusion

von Neumann stability: The new terms do not alter the results already obtained for the heat equation.

Different types of boundary conditions

So far: Dirichlet BCs

$$u(0,t) = \alpha(t)\,,$$
$$u(1,t) = \beta(t)\,,$$

Other BCs: Neumann BCs at $x = 0$ (or $x = 1$)

$$u_x(0,t) = \gamma(t) \quad (\text{or } u_x(1,t) = \gamma(t)),$$

is based on a Forward Difference approximation at $x = 0$ (or a Backward Difference approximation at $x = 1$).

Remark. We can also have mixed BCs (a mixture of Dirichlet and Neumann BCs).

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Forward Difference Approximation at $x = 0$

We make the approximation

$$u_x(0, t_n) \approx \frac{F_x}{h} u(0, t_n)$$

such that the BC implies $\frac{F_x}{h} w_0^n = \frac{w_1^n - w_0^n}{h} = \gamma(t_n) =: \gamma_n$, i.e.,

$$w_1^n - w_0^n = h\gamma_n \,.$$

In an explicit scheme: E.g. the FTCS scheme at $j = 1$

$$w_1^{n+1} = r w_0^n + (1 - 2r) w_1^n + r w_2^n$$

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In an implicit scheme: Provides two possibilities:

1. Add $w_1^n - w_0^n = h\gamma_n$ to the set of equations
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Method 1: Add w_0^{n+1} to the set of equations (e.g. in the case of the Crank-Nicolson schme, i.e., $\theta = 1/2$)

$$\mathbf{w}^{n+1} = \left[w_0^{n+1}, w_1^{n+1}, \ldots, w_{j-1}^{n+1}\right]'.$$

such that

$$S := \begin{pmatrix} -1 & 1 & 0 & \ldots & \\ -\frac{1}{2}r & 1+r & -\frac{1}{2}r & 0 & \ldots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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Advantage: Straightforward to apply

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The fictitious point method: Introduce a fictitious point

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outside the region/domain $[0, 1]$.

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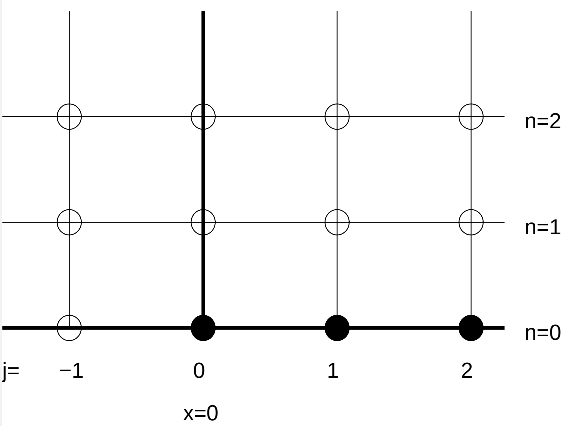

Step 1: A Central Difference approximation of $u_x(0, t_n)$ in space

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and after rewriting we have

$$w_1^n - w_{-1}^n = 2h\gamma_n \qquad (**)$$

Step 2: a) In an explicit scheme (FTCS). Eliminate the value w_{-1}^n, i.e., in the FTCS scheme

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Step 2: b) For an implicit scheme. Apply $(**)$ at times n and $n+1$, i.e., for the Crank-Nicolson scheme

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