Numerical Methods for PDEs

Multilevel Schemes, Convergence, and Lax Equivalence

(Lecture 9, Week 3)

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Outline

Multilevel Schemes: 1. The Richardson scheme

Multilevel schemes: are numerical schemes that involve more than 2 time levels. We consider subsequently only the heat equation.

1. The Richardson scheme: approximates u_t by the Central Difference operator $\frac{D_t}{k}u(x_j, t_n)$ such that with the usual $\frac{\delta_x^2}{h^2}$ in space we get

$$\frac{D_t}{k}w_j^n := \frac{w_j^{n+1} - w_j^{n-1}}{2k} = \frac{\delta_x^2}{h^2}w_j^n =: \frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{h^2},$$

which gives the Richardson scheme

$$w_j^{n+1} = w_j^{n-1} + 2r(w_{j-1}^n - 2w_j^n + w_{j+1}^n), \quad n \geq 1, j = 1, \ldots, J-1.$$

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Stability of the Richardson scheme

Claim: The Richardson scheme is *unstable for all* r.

Proof: Inserting $w_j^n = \xi^n e^{i\omega j}$ into the scheme gives

$$\xi^2 + 8r\sin^2\left(\frac{1}{2}\omega\right)\xi - 1 = 0\,.$$

Stability requires $|\xi_1|\,, |\xi_2| \leq 1$. Note that

$$(\xi - \xi_1)(\xi - \xi_2) = \xi^2 + b\xi + c$$

and hence

$$b = -(\xi_1 + \xi_2),\; c = \xi_1\xi_2\,.$$

Since $|c| = 1 = a$ and $|b| \neq 2$, we must have $|\xi_1|\,|\xi_2| = 1$ and either
(i) both ξ_i complex (Not the case since: $(\xi + i)^2 = \xi^2 + 2i\xi - 1$ and $b = 8r\sin^2(\omega/2)$ is not complex), or
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Solving the quadratic gives

$$\xi_{1,2} = \xi_{\pm} = -4r\sin^2\frac{\omega}{2} \pm \sqrt{1 + 16r^2\sin^4\frac{\omega}{2}}\,,$$

then if both roots are real, one will have $|\xi| > 1$. In fact

$$\xi_- \;=\; -p - \sqrt{1+p^2}, \quad p = 4r\sin^2\frac{\omega}{2} \geq 0$$

so clearly $|\xi_-| > 1$ when $p > 0$, and hence the Richardson sheme is *unstable* for *all* r.

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The Du Fort-Frankel scheme

Replace in the Richardson method the term w_j^n by the average $(w_j^{n+1} + w_j^{n-1})/2$, then we obtain the Du Fort-Frankel scheme

$$L_{k,h} w_j^n := \frac{w_j^{n+1} - w_j^{n-1}}{2k} - \left(\frac{w_{j-1}^n - w_j^{n-1} - w_j^{n+1} + w_{j+1}^n}{h^2} \right)$$

which for $r := k/h^2$ after re-arranging reads as follows

$$w_j^{n+1} = \frac{1-2r}{1+2r} w_j^{n-1} + \frac{2r}{1+2r} \left(w_{j-1}^n + w_{j+1}^n \right), \qquad \text{(DFS)}$$

Remark. This scheme is explicit but needs to be provided with $\mathbf{w}^1$ by a different scheme.

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Exercise: Show that the LTE of the Du Fort-Frankel scheme is

$$\text{LTE} = \left(r^2 - \frac{1}{12}\right) u_{tt}\, h^2 + \frac{k^2}{6} u_{ttt} + O(k^4, h^4, r^4 h^6)$$

i.e. it is **second order in time and space.**

Claim: The Du Fort-Frankel scheme is *unconditionally stable* for all $r > 0$.

Proof: By the *von Neumann stability* method one can easily show (Exercise) that the Du Fort-Frankel scheme leads to the following quadratic equation for the *amplification factor*

$$(1 + 2r)\xi^2 - 4r\xi \cos\omega + 2r - 1 = 0,$$

whose roots are

$$\xi_\pm = \frac{2r\cos\omega \pm \sqrt{1 - 4r^2 \sin^2\omega}}{1 + 2r}.$$

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We discuss now two different cases w.r.t. the discriminant:

(i) $4r^2 \sin^2 \omega \leq 1$, so both roots are real. It follows that

$$\xi_+ = \frac{2r\cos\omega + \sqrt{1 - 4r^2\sin^2\omega}}{1+2r} \leq \frac{2r\cos\omega + 1}{1+2r} \leq 1,$$

since $\cos\omega \leq 1$ for all ω, and moreover (since $\cos\omega \geq -1$ for all ω)

$$\xi_+ \geq \frac{-2r + \sqrt{1 - 4r^2\sin^2\omega}}{1+2r} \geq \frac{-2r}{1+2r} > -1,$$

i.e. $-1 < \xi_+ \leq 1$. Similarly, it holds that $-1 \leq \xi_- < 1$, so in this case both roots satisfy $|\xi| \leq 1$.

(ii) $4r^2 \sin^2 \omega > 1$, so both roots of the quadratic are complex, i.e. $\xi_\pm = \alpha \pm i\beta$, where

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$$|\xi|^2 = \frac{|2r-1|}{1+2r} \leq 1$$

for all $r > 0$, and so $|\xi| \leq 1$ for any r.

This proves the claim. ∎

Comparison of methods:

Scheme	Comments	Stability	LTE
Du Fort-Frankel	explicit, but different scheme required for $\vec{w}^1$	$\forall\, r$	$O(r^2h^2, h^2, k^2)$. Second order if $k = O(h^2)$, fourth order if $k = h^2/\sqrt{12}$
Crank-Nicolson ($\theta = 1/2$)	implicit: need to solve a tridiagonal system (not too bad)	$\forall\, r$	$O(h^2, k^2)$. Second order in space and time separately
FTCS ($\theta = 0$)	explicit	$\forall r \leq 1/2$	$O(h^2, k)$. Second order if $k = O(h^2)$, fourth order if $k = h^2/6$

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Convergence

Basic idea: For a difference operator $L_{k,h}$ that approximates $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ and for numerical and exact solutions w_j^n and u, respectively, i.e., solutions of $L_{k,h}w_j^n = 0$ and $Lu = 0$, respectively, we say that a numerical scheme $L_{k,h}$ *converges* if $w_j^n \to u$ for $h, k \to 0$.

Pointwise convergence: Fix $x^* \in (0, 1)$ and $t^* > 0$. We are interested in $h, k \to 0$ for $x^* = jh$ and $t^* = nk$ fixed. Hence, we can write $w_j^n = w_{x^*/h}^{t^*/k}$.

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Lax Equivalence Theorem

Theorem: If the linear finite difference scheme is consistent (i.e., its $\mathrm{LTE} \to 0$ as $h, k \to 0$), then

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Examples of more general parabolic PDEs

- **Reaction-Diffusion equations** The general form is

$$u_t = \kappa u_{xx} + f(x,t,u)$$

  where f represents the reaction term.
  Note: r becomes κr, hence the FTCS scheme is unstable for $\kappa r > 1/2$. The reaction term is approximated by $f(x_j, t_n, w_j^n)$ at time level n. If $f(x,t,u)$ is nonlinear in u, and if we use an implicit scheme, then we will end up with a set of *nonlinear* equations for w_j^{n+1} at each time level.
- **Linear equations with *varying* coefficients**
  A typical equation is

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u$$

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- **The Black-Scholes equation**
  is a linear equation with variable coefficients. It describes the value of an *option* to buy shares at time T at the price E. If $S(T)$ is the value of the share price at $t = T$, and if $S(T) > E$, buy them (exercise the option), if $S(T) \leq E$, don't buy (no profit). What is the value of this option $V(t, S)$ at $t = 0$? It satisfies the Black-Scholes PDE

$$V_t + \rho S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss} - \rho V = 0, \quad t \in [0, T]$$

  where ρ is the interest rate, σ is the share volatility, and S is the share price. The boundary conditions are $V(0, t) = 0$ and $\lim_{S\to\infty} V(S, t)/S = 1$, since ($V \sim S - E$). The final condition is $V(S, T) = \max(S - E, 0)$, E given. We know $S = S_0$ at $t = 0$ (i.e. now) and want to work out $V(S_0, 0)$. We approximate $V(S_j, t_n) \approx W_j^n$.

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Approximating for the terms in the usual way we get for example for the FTCS scheme:

$$\frac{W_j^{n+1} - W_j^n}{k} + \rho S_j \frac{W_{j+1}^n - W_{j-1}^n}{2\Delta S} + \frac{1}{2}\sigma^2 S_j^2 \frac{W_{j-1}^n - 2W_j^n + W_{j+1}^n}{\Delta S^2} - \rho W_j^n = 0$$

However the method of solutions is a little different, we solve this starting at $t = T$ and working **backwards in time** to get to $t = 0$. Then we see if the computed value $V(S_0, 0)$ is higher or lower than the price being asked for the option.